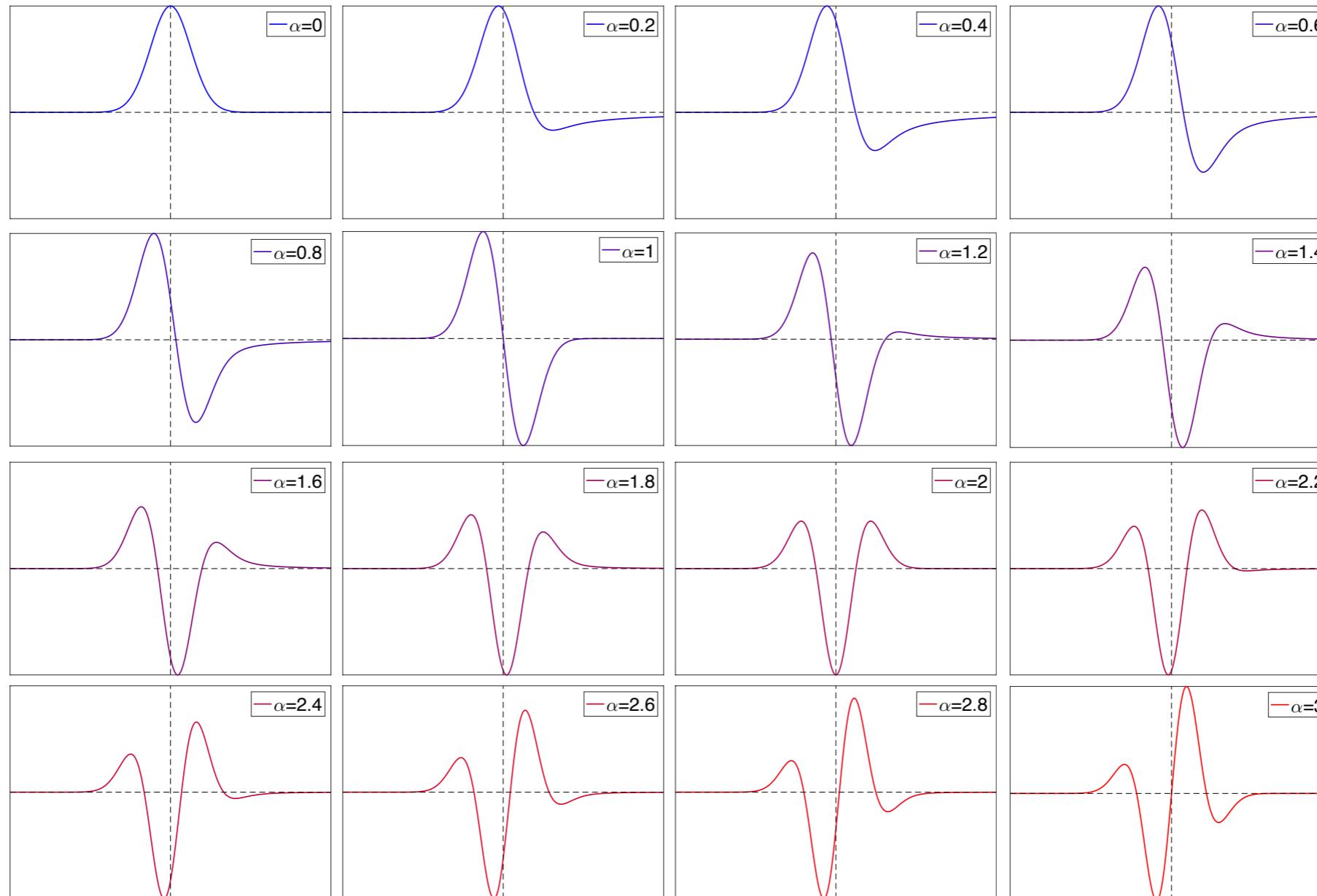


# **Signal and Image Processing**

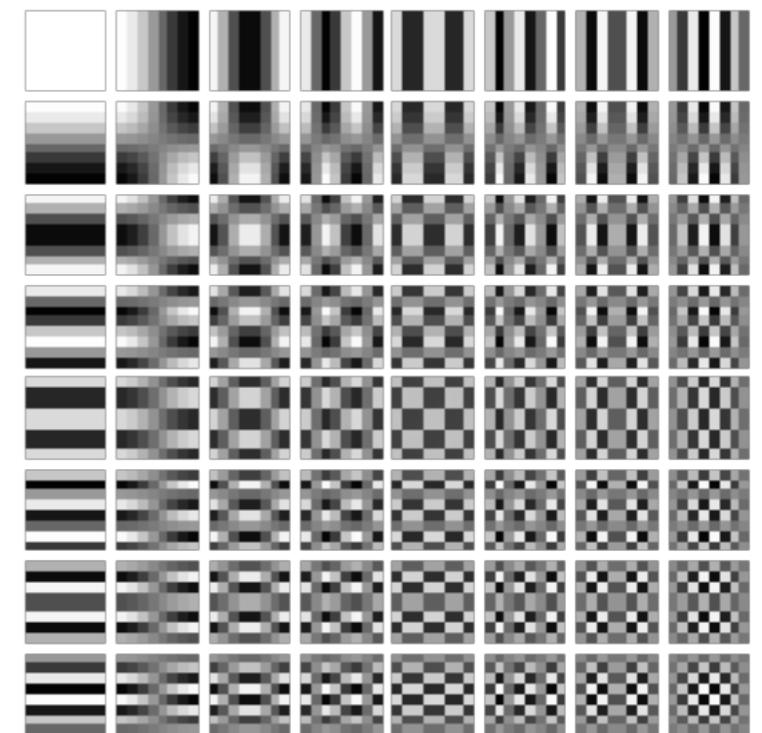
Fourier transform:  $\mathcal{F}(f)(\omega) \stackrel{\text{def.}}{=} \int_{\mathbb{R}} f(x)e^{-i\omega x}dx$

Fractional derivative:  $\mathcal{F}(f^{(\alpha)}) \stackrel{\text{def.}}{=} (i\omega)^{\alpha} \mathcal{F}(f)(\omega)$



DCT-1:	$\cos jk \frac{\pi}{N-1}$	(divide by $\sqrt{2}$ when $j$ or $k$ is 0 or $N - 1$ )
DCT-2:	$\cos \left(j + \frac{1}{2}\right) k \frac{\pi}{N}$	(divide by $\sqrt{2}$ when $k = 0$ )
DCT-3:	$\cos j \left(k + \frac{1}{2}\right) \frac{\pi}{N}$	(divide by $\sqrt{2}$ when $j = 0$ )
DCT-4:	$\cos \left(j + \frac{1}{2}\right) \left(k + \frac{1}{2}\right) \frac{\pi}{N}$	

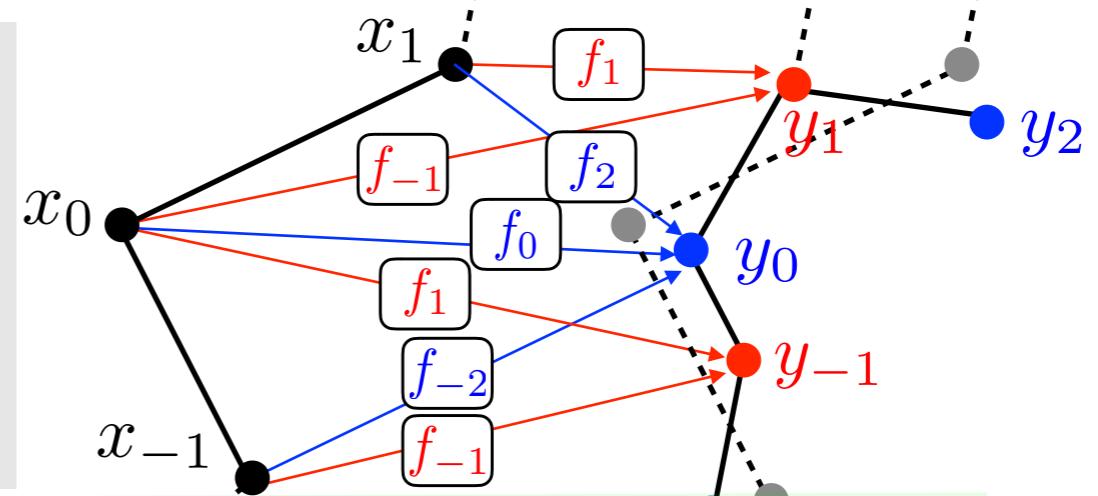
**The discrete case has a new level of variety and complexity, often appearing in the boundary conditions [G. Strang - SIAM review, 1999]**



Refinement scheme:

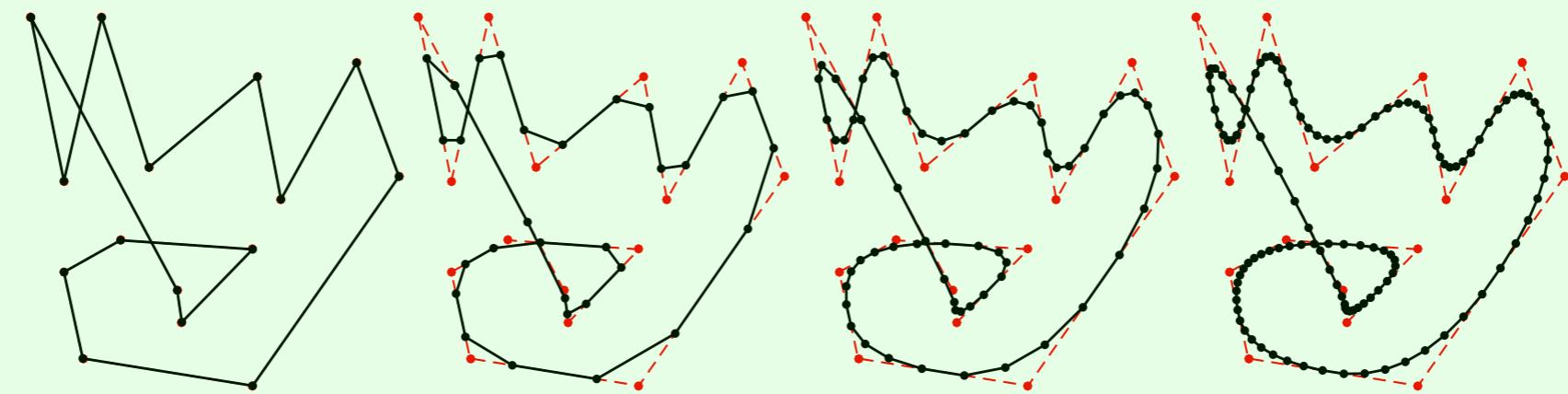
$$y_{2k} = \sum_i f_{2i} x_{k+i}$$

$$y_{2k+1} = \sum_i f_{2i+1} x_{k+i}$$



Approximating

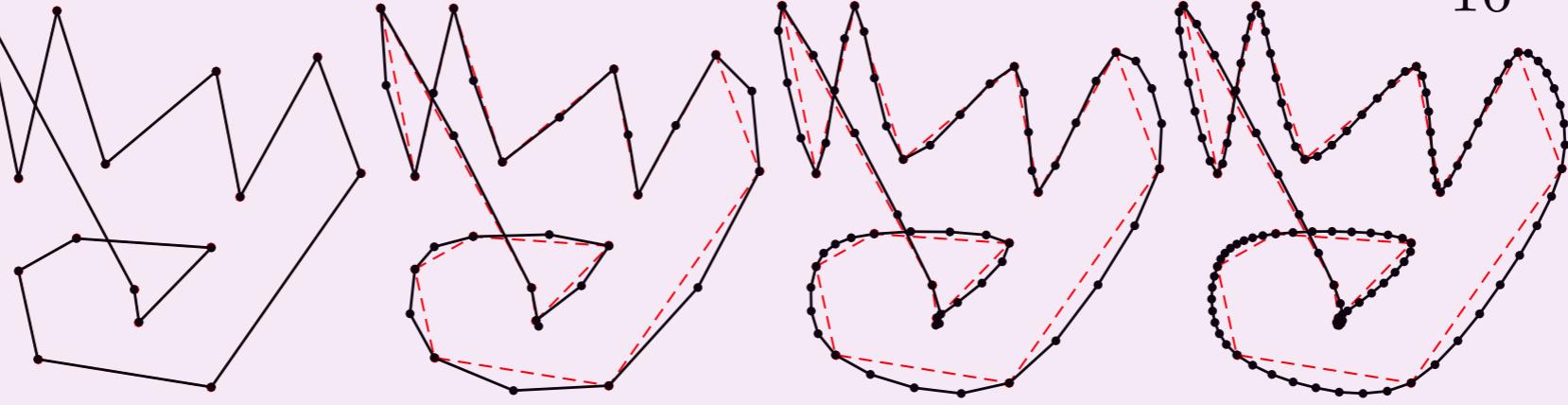
$$f = \frac{1}{4}(1, 3, 3, 1)$$

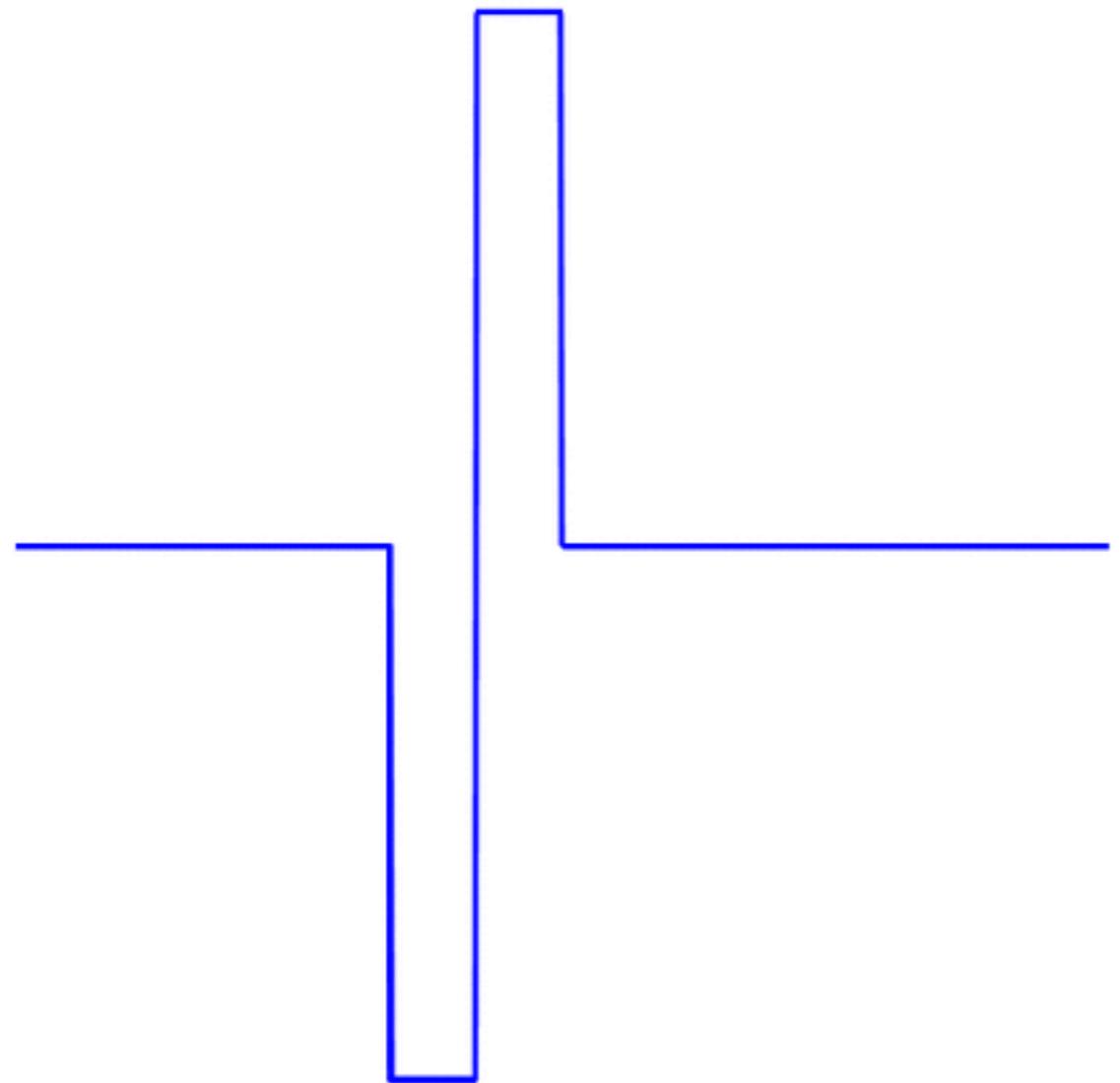


Interpolating

$$f = (-w, 0, 1/2 + w, 1, 1/2 + w, 0, -w)$$

$$w = \frac{1}{16}$$

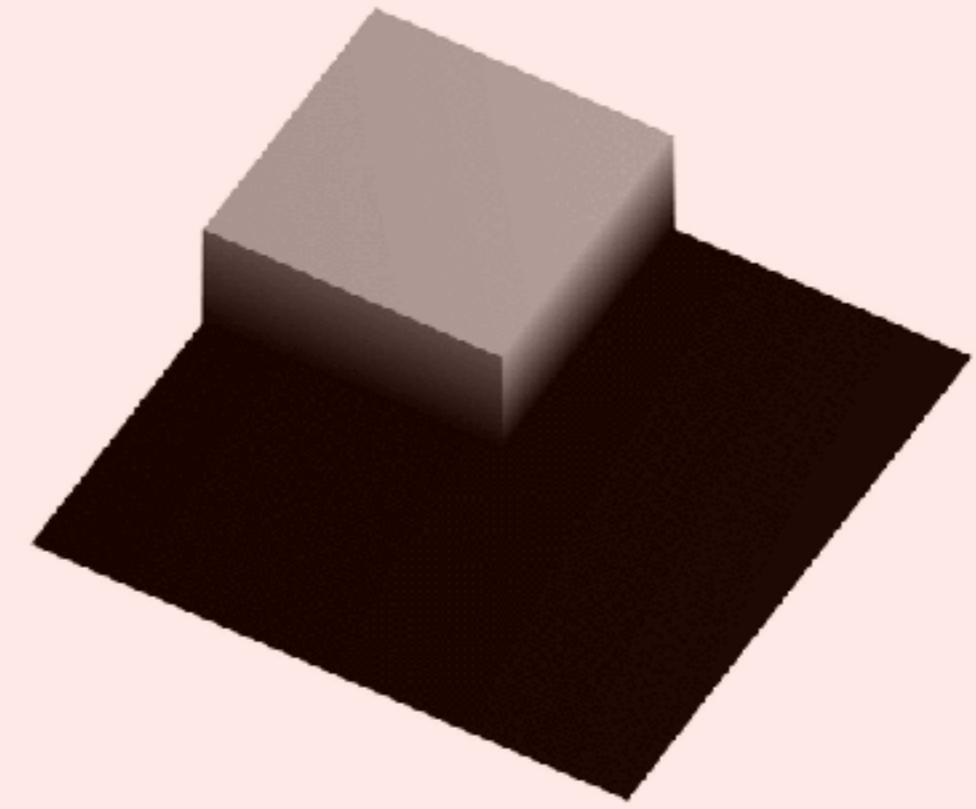
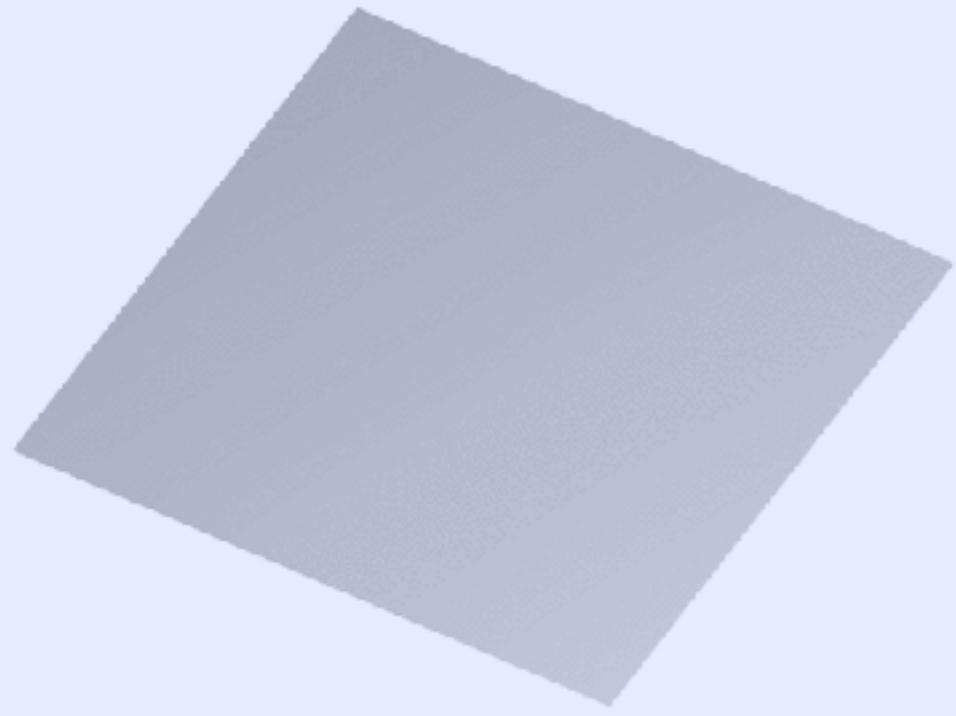
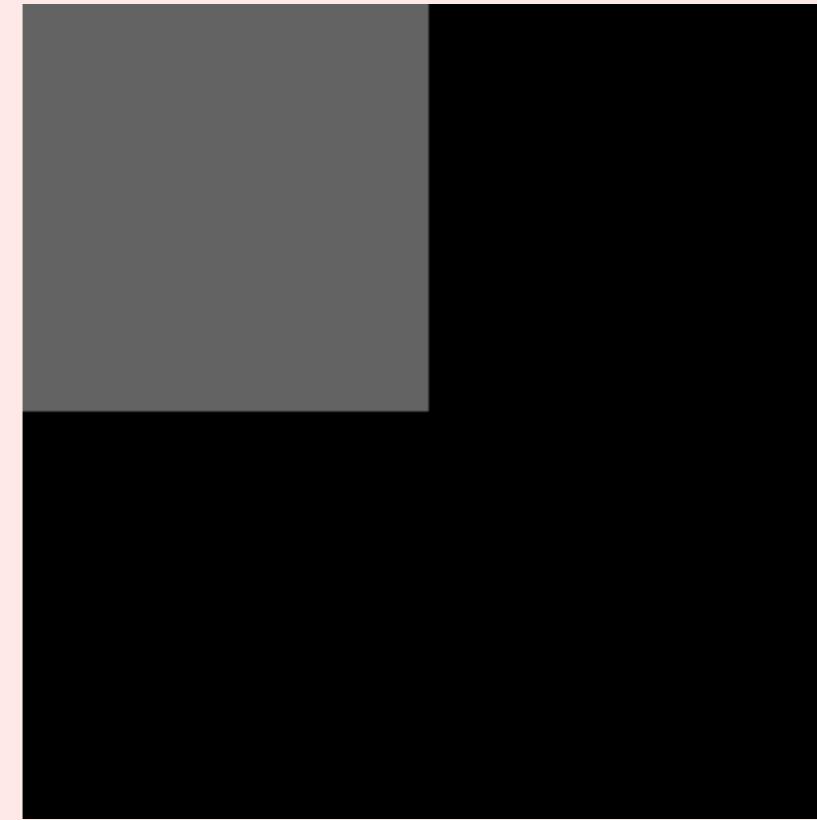




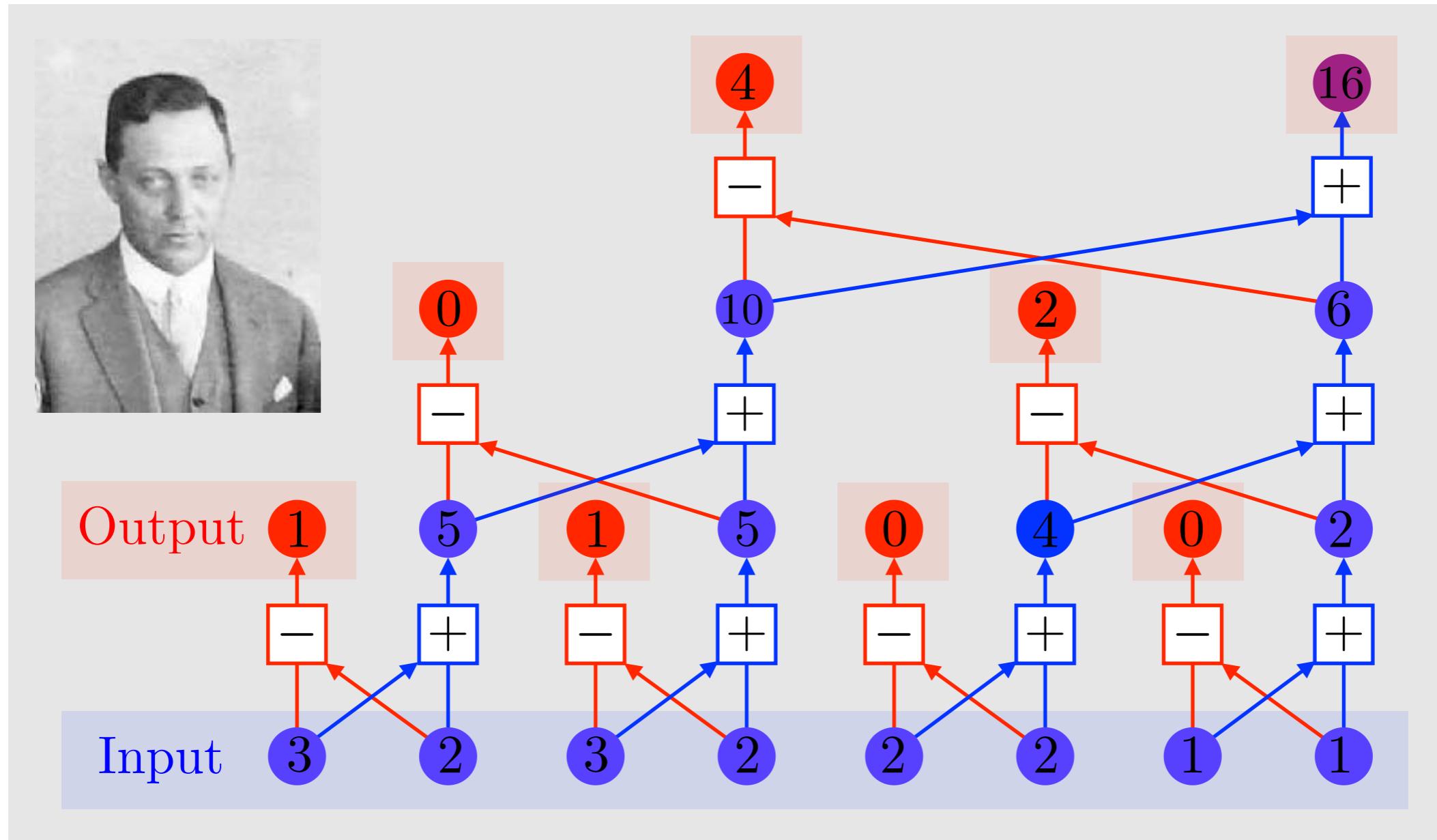
Linear (uniform)



Non-linear (adaptive)



$$(3, 2, 3, 2, 2, 2, 1, 1) \xrightarrow[\text{transform}]{\text{Haar}} (1, 1, 0, 0, 0, 2, 4, 16)$$





Joseph Walsh



Hans Rademacher



Jacques Hadamard



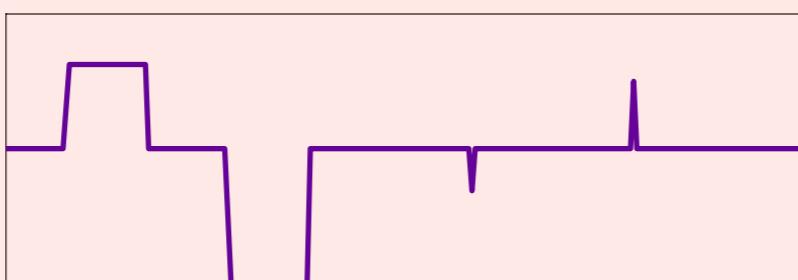
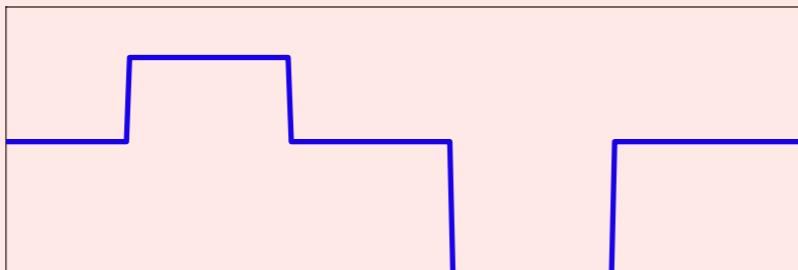
Alfréd Haar

Walsh:  $n \log(n)$  operations.

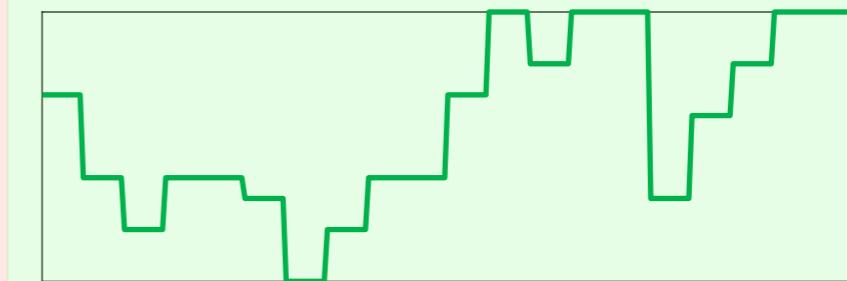
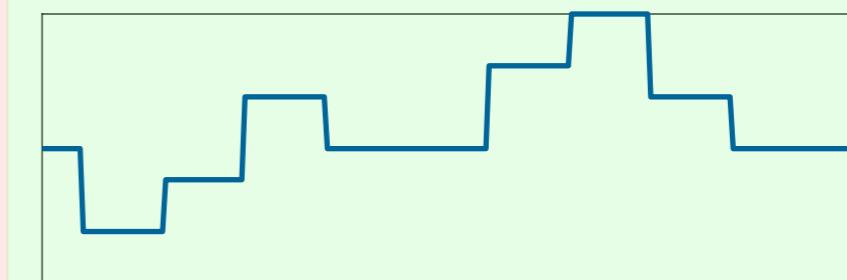
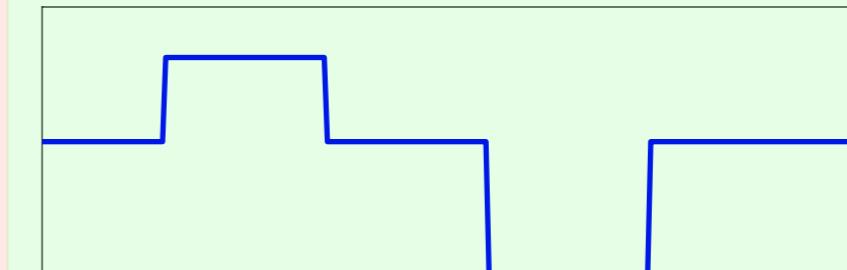
$$W(f) = (W(f_{1:\frac{n}{2}}) + W(f_{\frac{n}{2}+1:n}), W(f_{1:\frac{n}{2}}) - W(f_{\frac{n}{2}+1:n}))$$

Haar:  $2n$  operations.

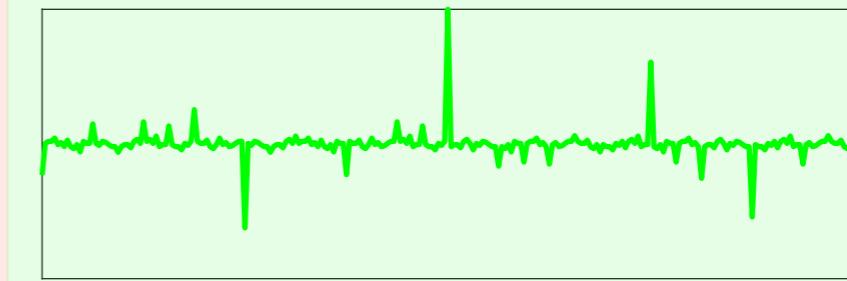
$$H(f) = (H(f_{1:2:n-1} + f_{2:2:n}), f_{1:2:n-1} - f_{2:2:n})$$

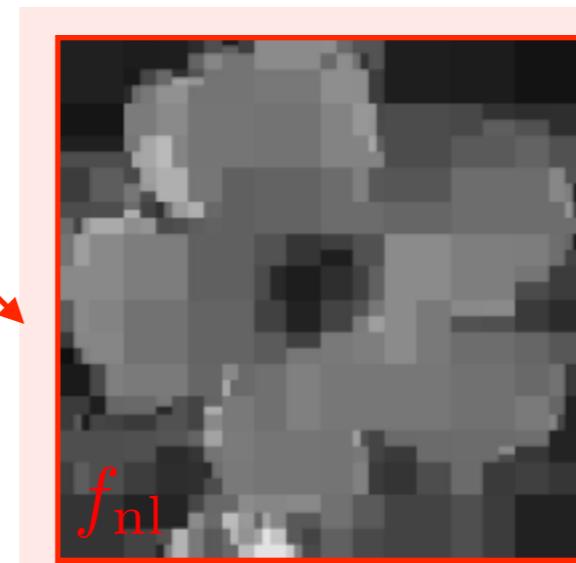
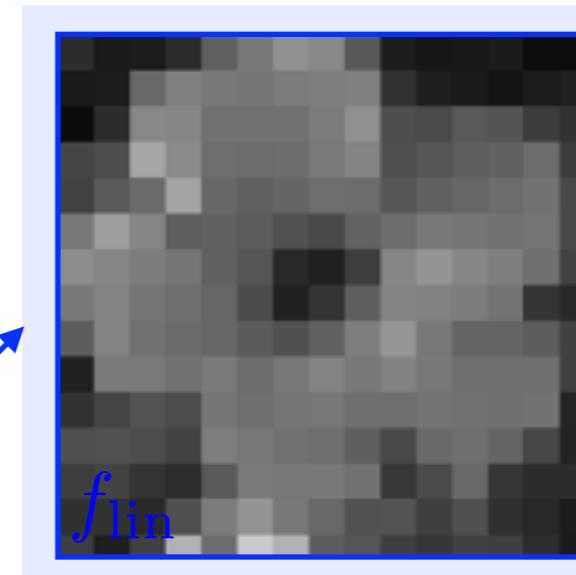


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• • •





Orthonormal basis  $(\psi_m)_m$ .

$$f_{\text{lin}} = \sum_{m=1}^n \langle f, \psi_m \rangle \psi_m$$

$$f_{\text{nl}} = \sum_{i=1}^n \langle f, \psi_{m_i} \rangle \psi_{m_i}$$

$$|\langle f, \psi_{m_1} \rangle| \geq |\langle f, \psi_{m_2} \rangle| \geq \dots$$

Alphabet  $K$ , probabilities  $p = (p_k)_{k \in K}$ .

Binary code:  $k \in K \mapsto c_k \in \{0, 1\}^{|c_k|}$ .

Average code length:  $L(c) = \sum_k p_k |c_k|$

$$T = \text{Huffman}(K, p)$$

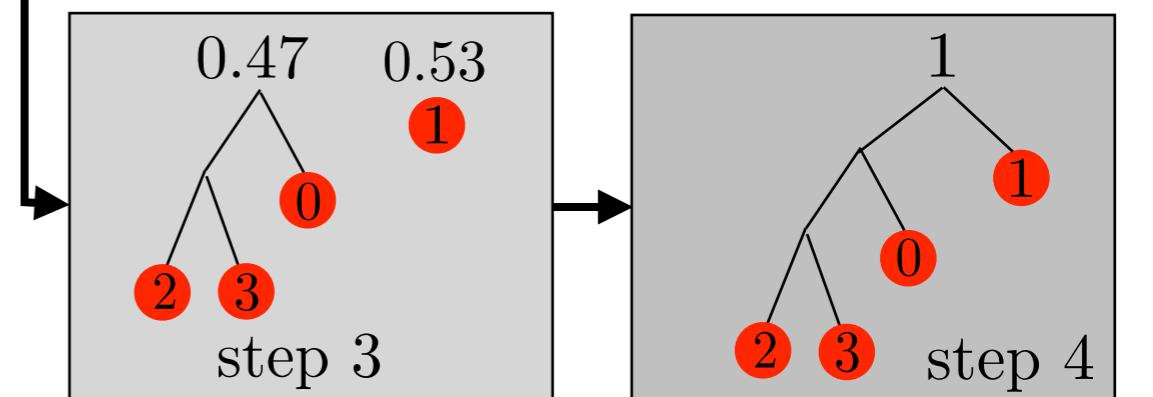
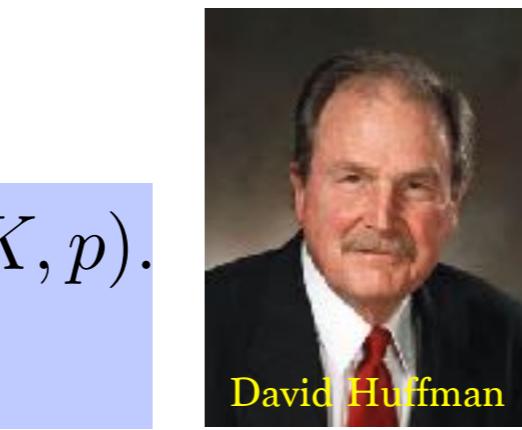
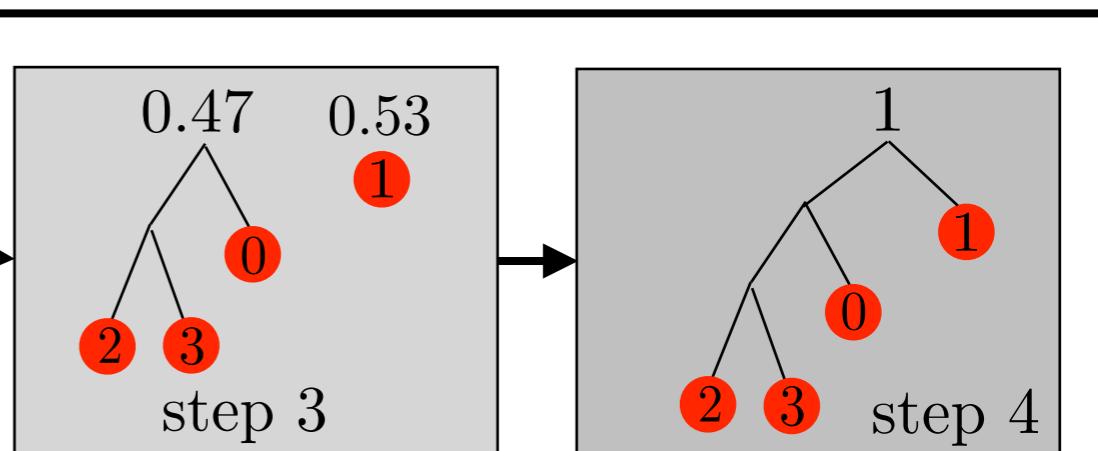
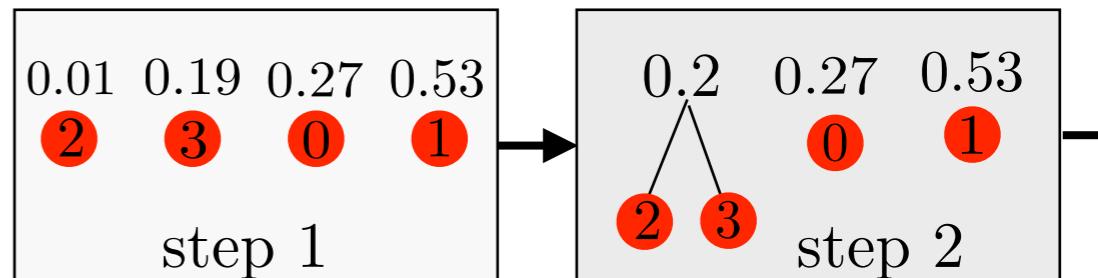
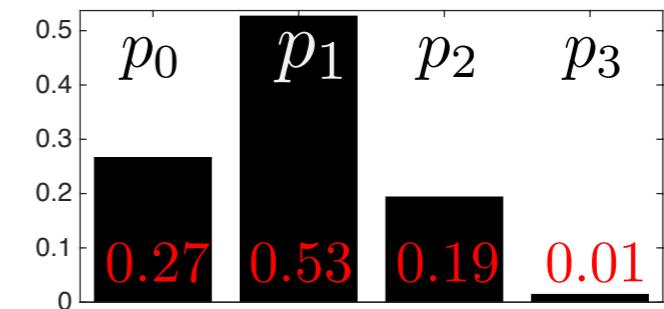
- If  $|K| = 1$ ,  $T = \{\cdot\}$ .
- If  $|K| > 1$ , sort  $p_a \leq p_b \leq \dots$

$$K' \stackrel{\text{def.}}{=} (K \setminus \{a, b\}) \cup \{z\}$$

$$p'_k = \begin{cases} p_a + p_b & \text{si } k = z, \\ p_k & \text{otherwise.} \end{cases}$$

$$T' \stackrel{\text{def.}}{=} \text{Huffman}(K', p')$$

$T$ : add  $(a, b)$  under  $z$ .



$c_0 = 01$     $c_2 = 000$   
 $c_1 = 1$     $c_3 = 001$

code words

*Theorem:* let  $c_H$  the code of  $\text{Huffman}(K, p)$ .

For any prefix code  $c$ ,  $L(c) \geq L(c_H)$

Continuous

Acquisition  
→ cones of the retina

Discrete

Wavelet transform  
→ simple cells in V1

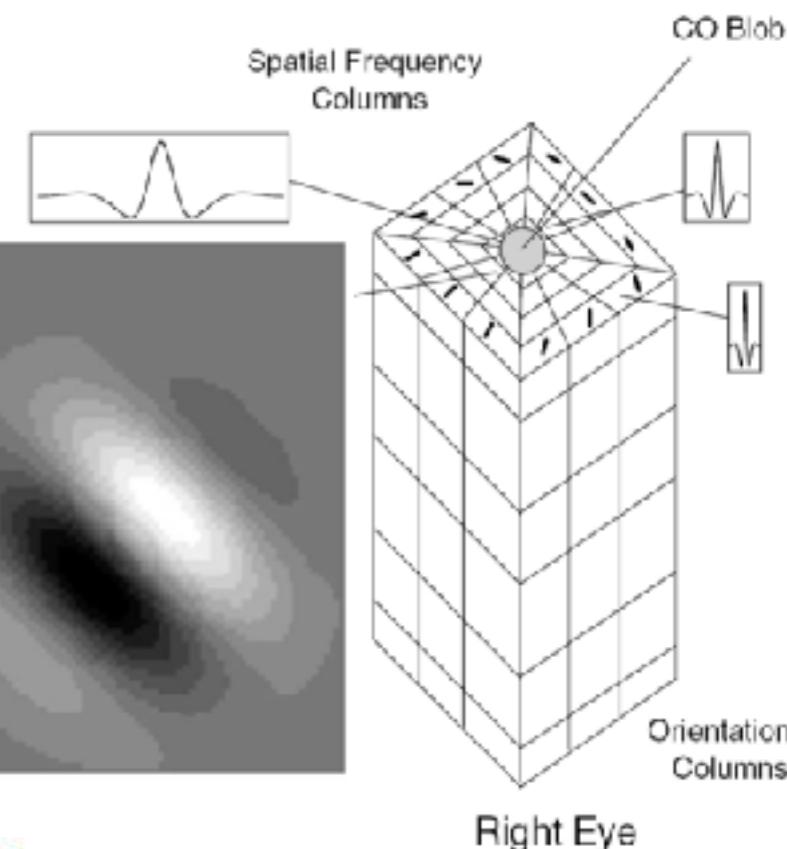
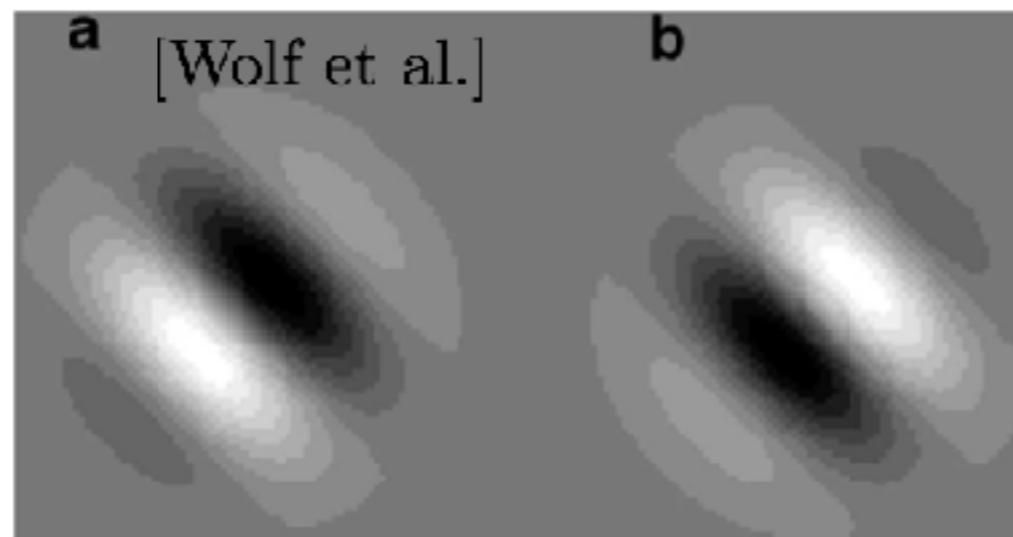
Multiscale



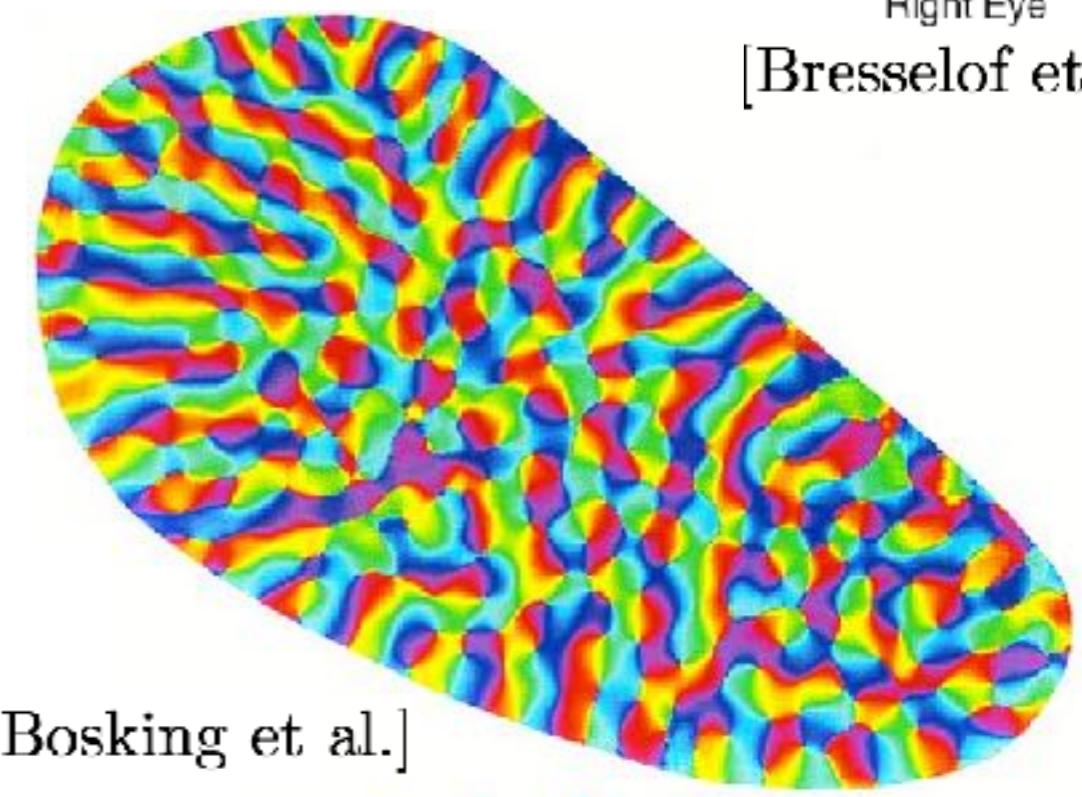
David Hubel



Torsten Wiesel



[Bressloff et al.]



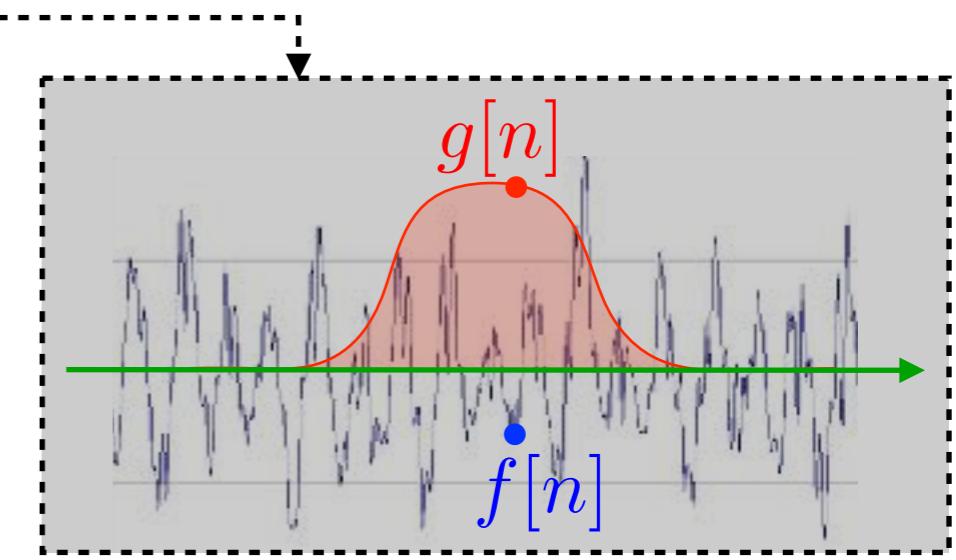
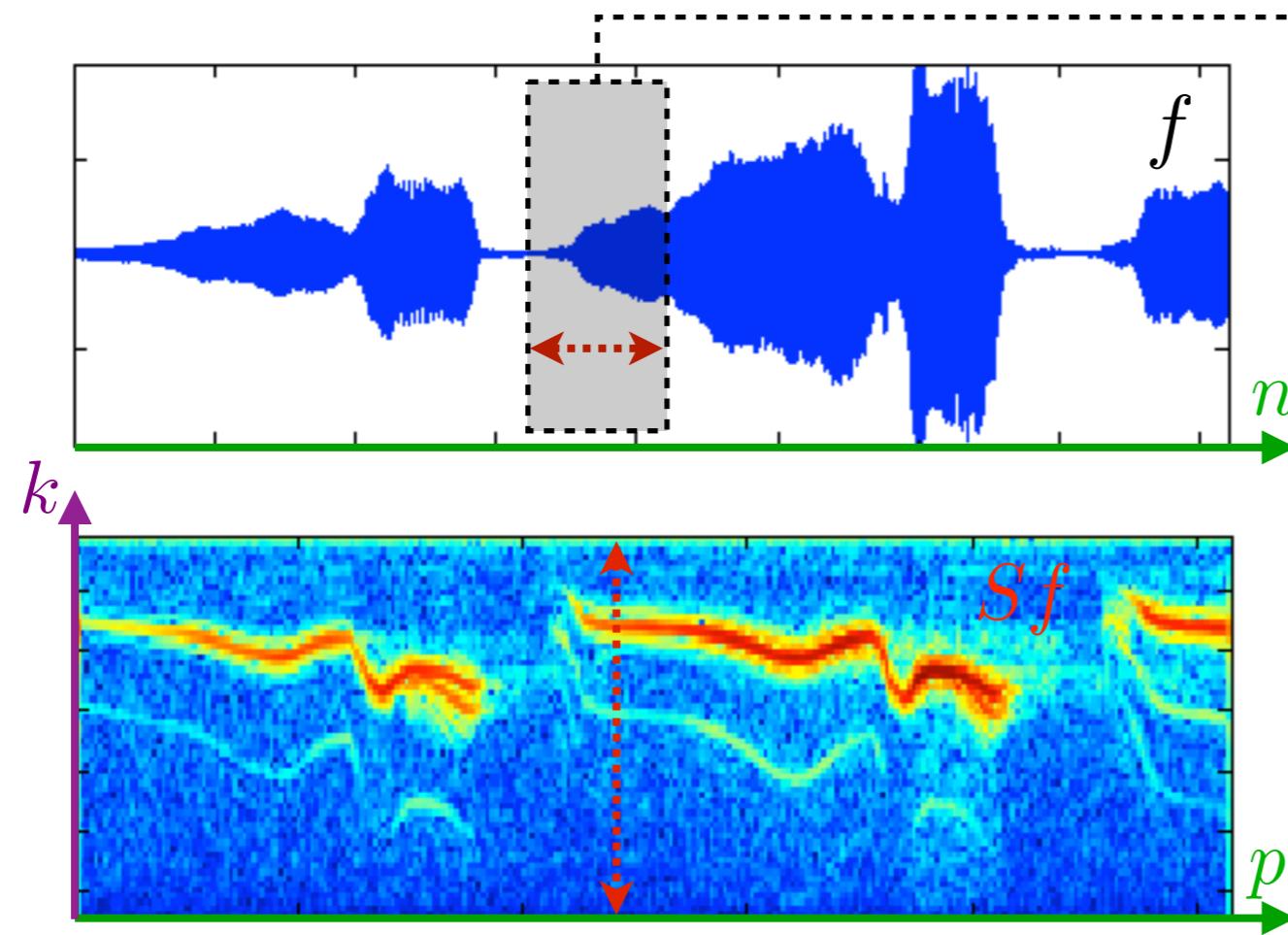
[Bosking et al.]

Spectrogramm / Short-time Fourier transform:

$$Sf[k, p] = Q^{-1/2} \sum_n f[n] g[\Delta_x p - n] e^{-\frac{2i\pi}{Q} kn}$$

frequency
time

signal  $f$   
window  $g$

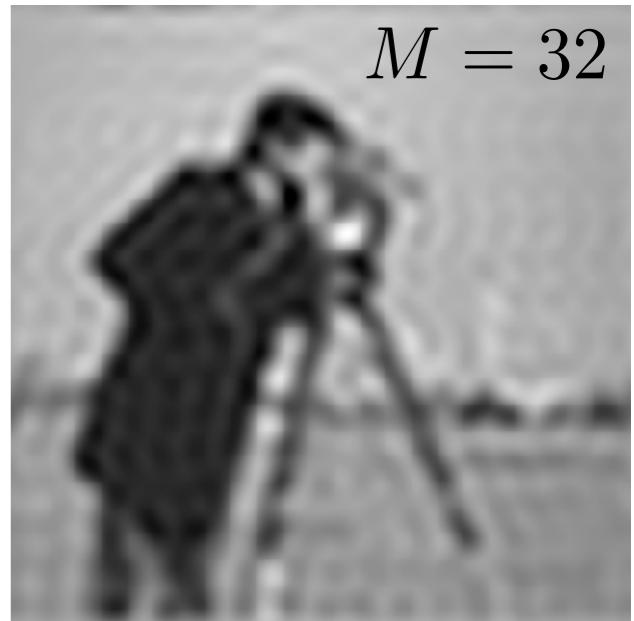
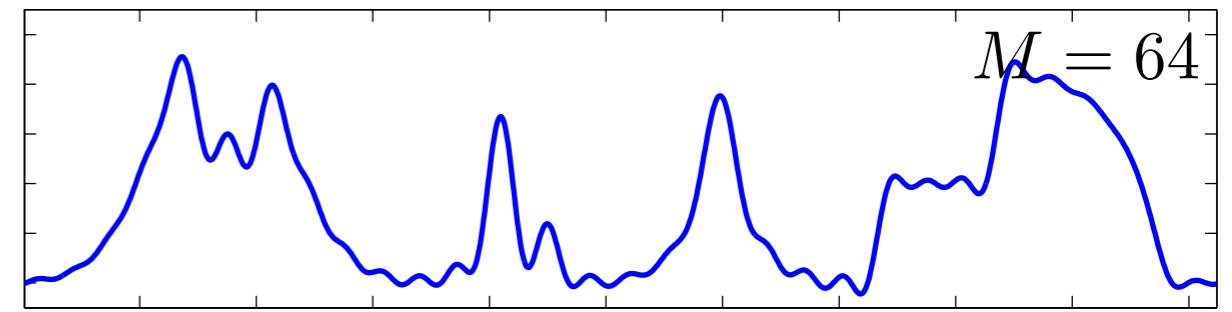
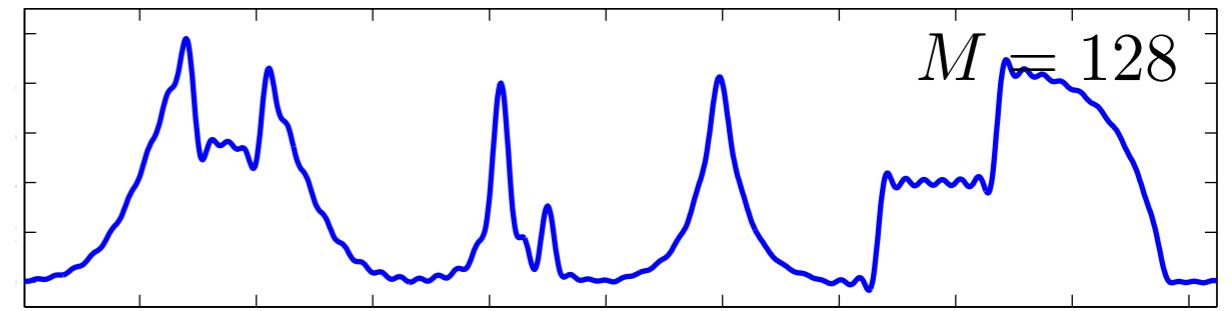
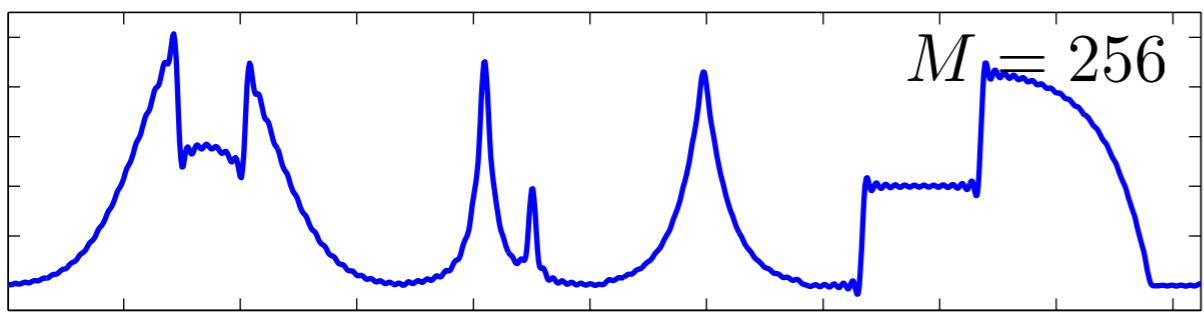
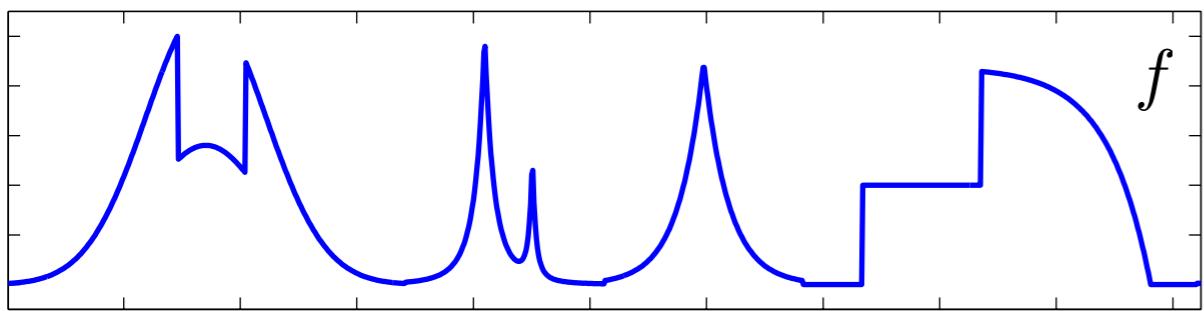
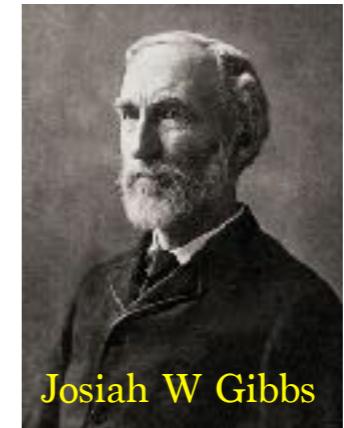


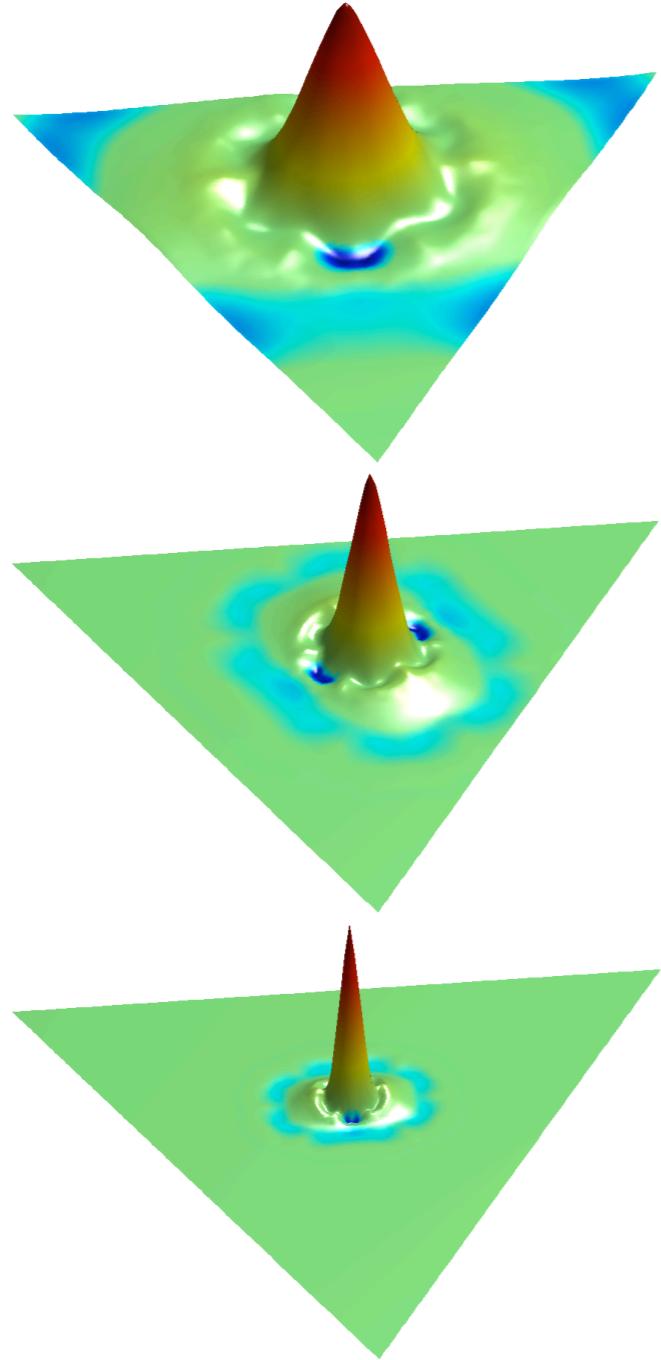
Computation:  
 $N$  FFTs of size  $Q$   
 $O(NQ \log(Q))$  operations.

Fourier atoms:  $e_m \stackrel{\text{def.}}{=} e^{2i\pi\langle m, x \rangle}$

Frequency  $m$ .

Linear Fourier approximation:  $f_M \stackrel{\text{def.}}{=} \sum_{|m| \leq M/2} \langle f, e_m \rangle e_m$



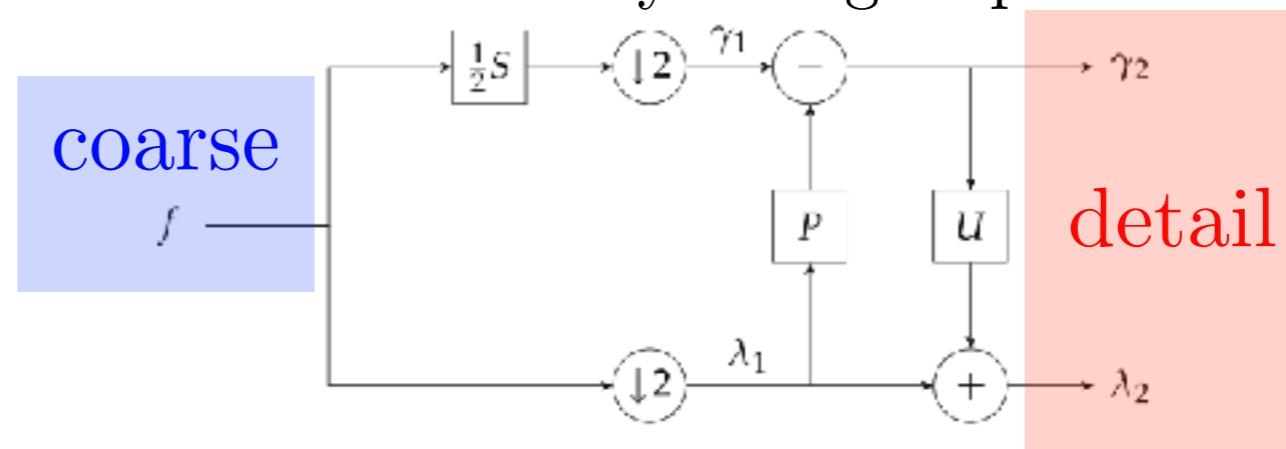


Wim Sweldens



Peter Schröder

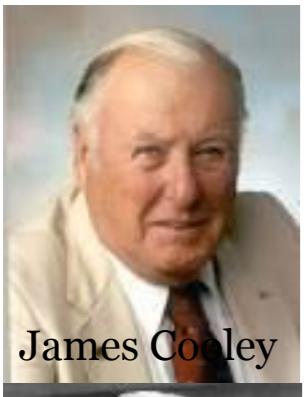
Elementary lifting step:



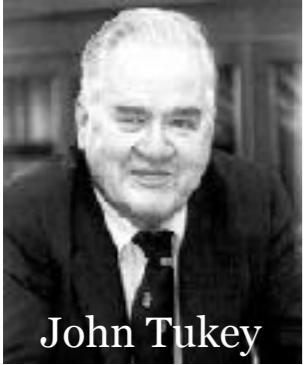


Carl Friedrich Gauss

...



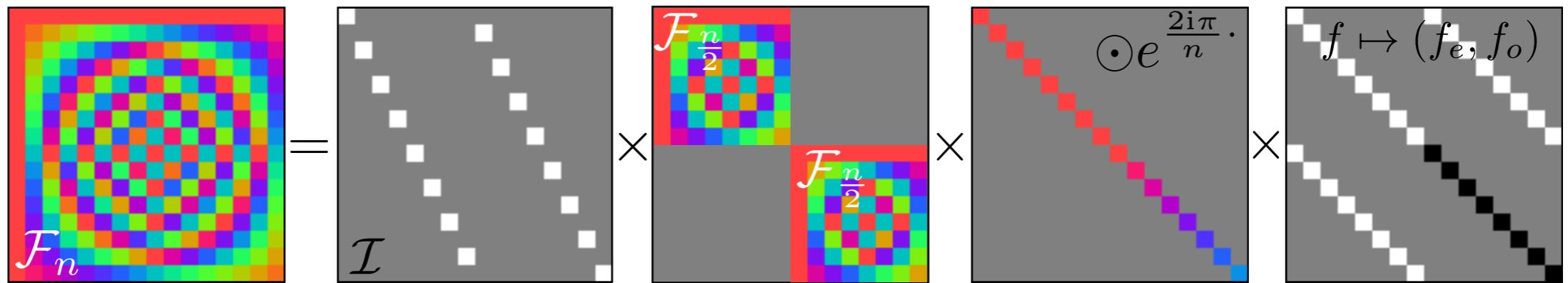
James Cooley



John Tukey

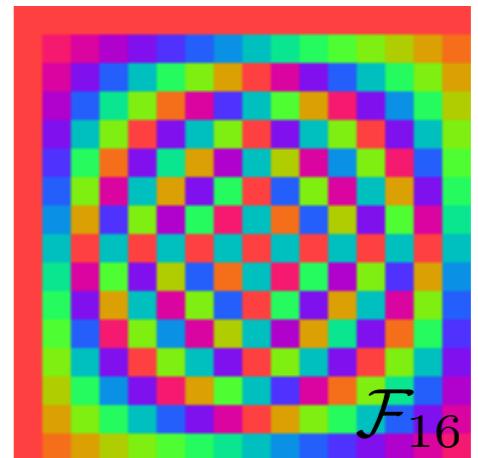
Discrete Fourier transform:  $\mathcal{F}_n(f)_\ell \stackrel{\text{def.}}{=} \sum_{k=0}^{n-1} f_k e^{\frac{2i\pi}{n} k\ell}$

FFT (1 step):  $\mathcal{F}_n(f) = \mathcal{I}(\mathcal{F}_{\frac{n}{2}}(f^+), \mathcal{F}_{\frac{n}{2}}(f^- \odot e^{\frac{2i\pi}{n} \cdot}))$   
 $f^\pm \stackrel{\text{def.}}{=} f_\cdot \pm f_{\cdot + \frac{n}{2}}$



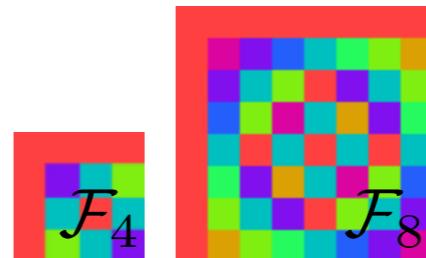
$$\text{Time}(\mathcal{F}_n) = 2\text{Time}(\mathcal{F}_{\frac{n}{2}}) + cn \implies \text{Time}(\mathcal{F}_n) = cn \log_2(n)$$

Discrete Fourier matrix:  $\mathcal{F}_n \stackrel{\text{def.}}{=} (e^{\frac{-2i\pi}{n} k\ell})_{k,\ell}$



Convolution:

$$(f \star h)_i \stackrel{\text{def.}}{=} \sum_j f(j)h(i-j)$$



Convolution matrix:  $C_h = (h(i - j \bmod n))_{i,j}$  i.e.  $C_h(f) = f \star h$

Fourier convolution theorem:  $\mathcal{F}_n(f \star h) = \mathcal{F}_n(f) \odot \mathcal{F}_n(h)$

$$\begin{matrix} \text{A grayscale checkerboard pattern} \\ C_h \end{matrix} = \begin{matrix} \mathcal{F}_n \end{matrix} \times \begin{matrix} \text{A diagonal matrix with colored entries} \\ \text{diag}(\mathcal{F}_n h) \end{matrix} \times \begin{matrix} \mathcal{F}_n^* \end{matrix}$$

The diagram illustrates the Fourier convolution theorem. It shows that the convolution matrix  $C_h$  is equal to the product of three matrices: the Discrete Fourier Matrix  $\mathcal{F}_n$ , the diagonal matrix  $\text{diag}(\mathcal{F}_n h)$  (which represents the Fourier transform of the kernel), and the inverse transpose of the Discrete Fourier Matrix,  $n^{-1} \mathcal{F}_n^*$ .

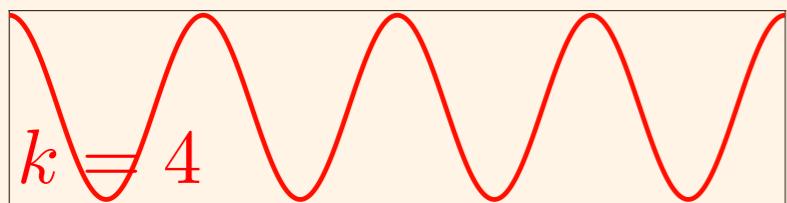
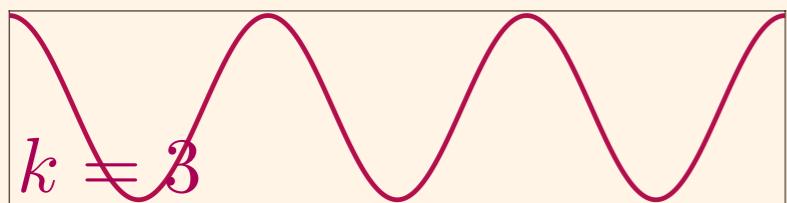
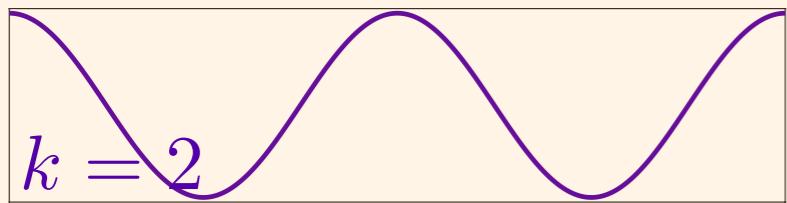
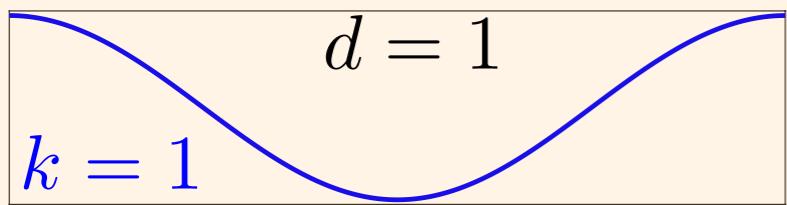
Continuous

$$e_{\mathbf{k}}(\mathbf{x}) = e^{2i\pi \langle \mathbf{k}, \mathbf{x} \rangle}$$

$$\mathbf{k} \in \mathbb{Z}^d \quad \mathbf{x} \in [0, 1]^d$$

Orthogonal for:

$$\int_{[0,1]^d} f(\mathbf{x})g(\mathbf{x}) d\mathbf{x}$$



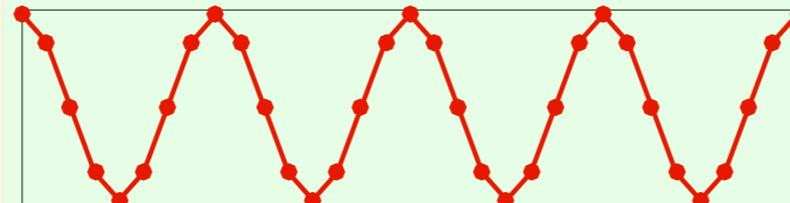
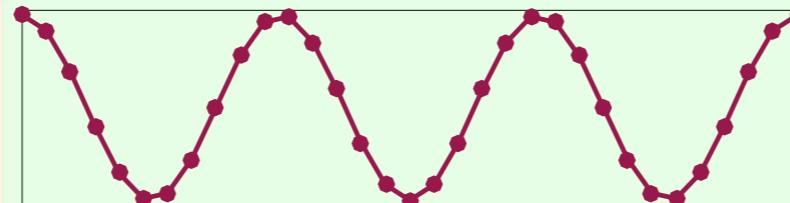
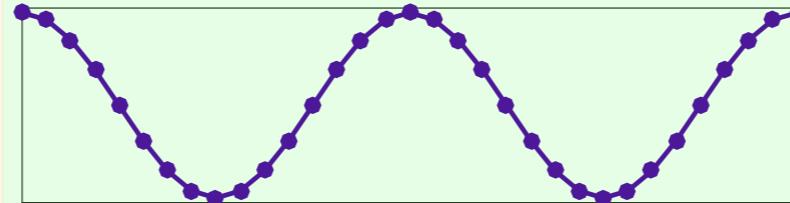
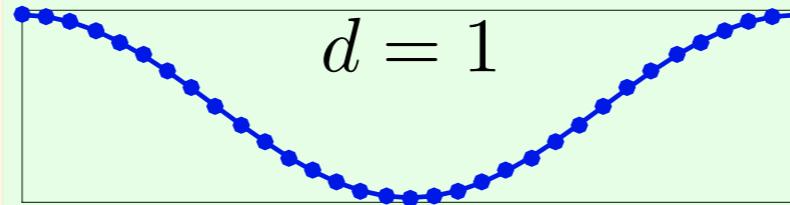
Discrete

$$e_{\mathbf{k}}(\mathbf{x}) = e^{\frac{2i\pi}{N} \langle \mathbf{k}, \mathbf{x} \rangle}$$

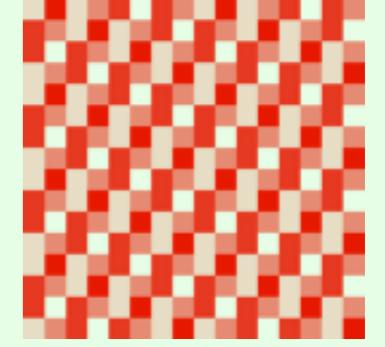
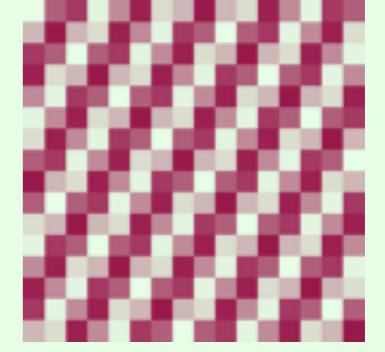
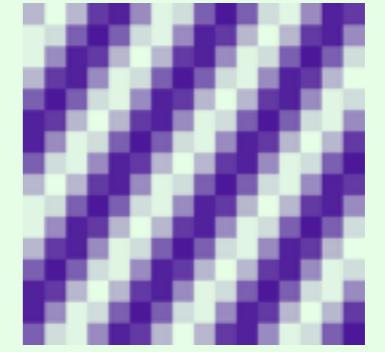
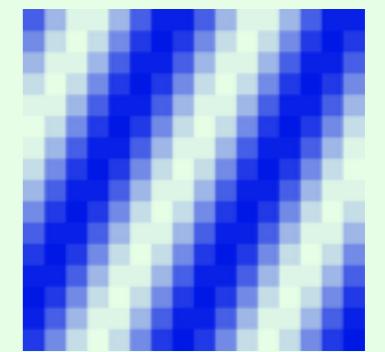
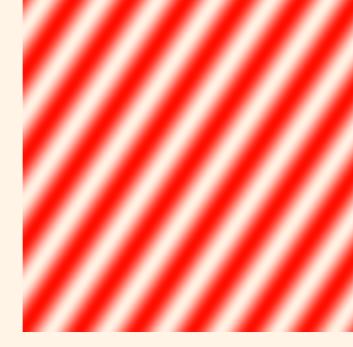
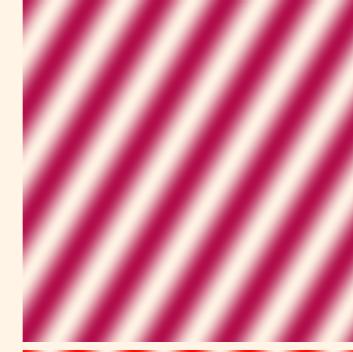
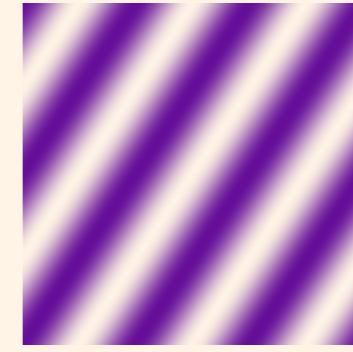
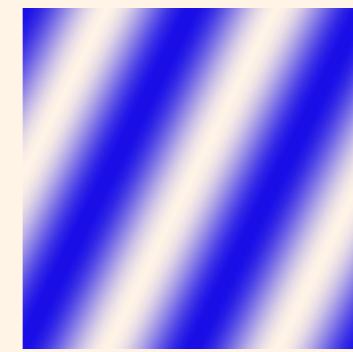
$$\mathbf{k}, \mathbf{x} \in \{0, \dots, N-1\}^d$$

Orthogonal for:

$$\sum_{x=0}^{N-1} f(\mathbf{x})g(\mathbf{x})$$



$d = 2$



1-D discrete Fourier basis:  $e_k[n] \stackrel{\text{def.}}{=} e^{\frac{2i\pi}{N} kn}$

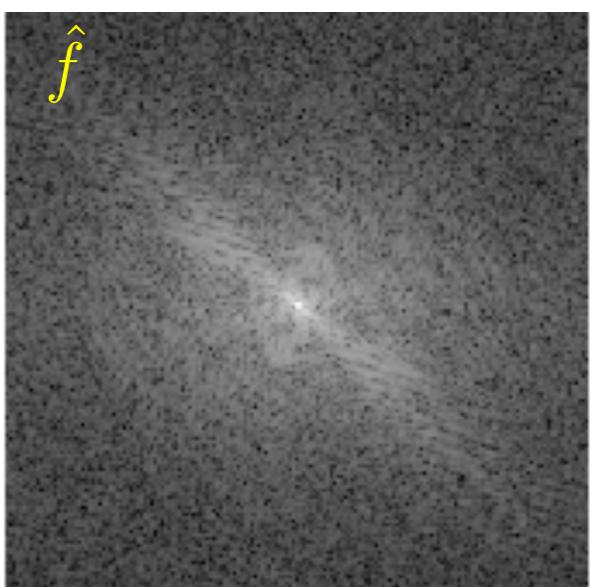
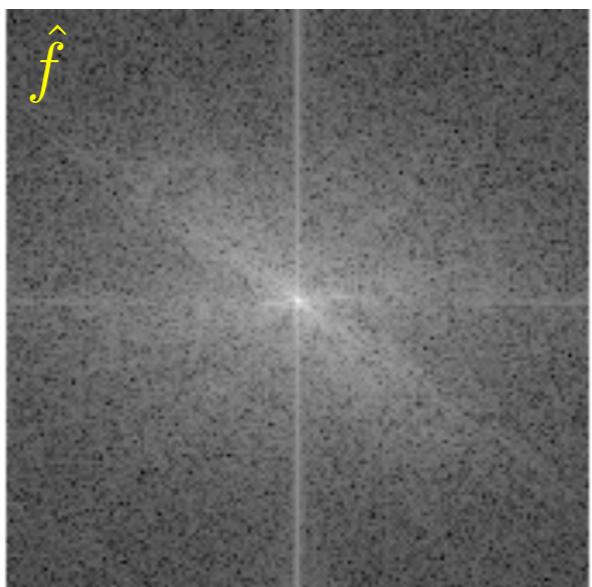
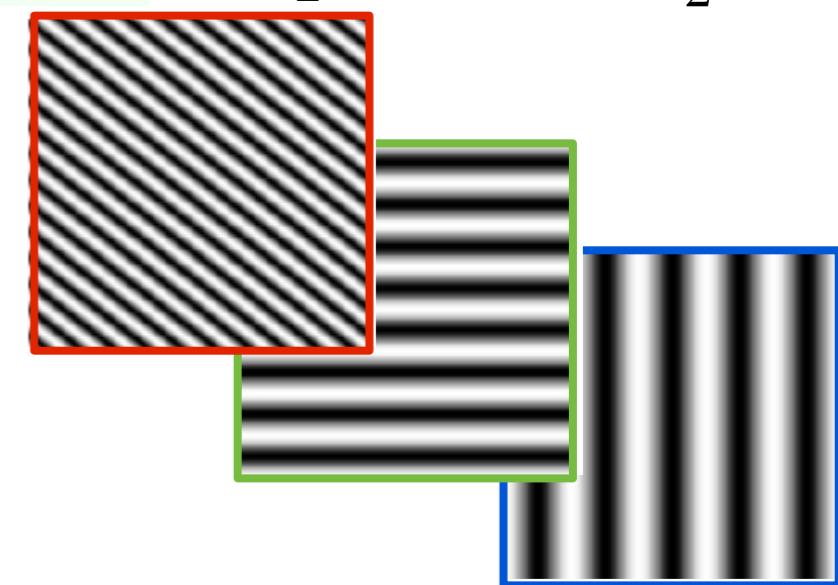
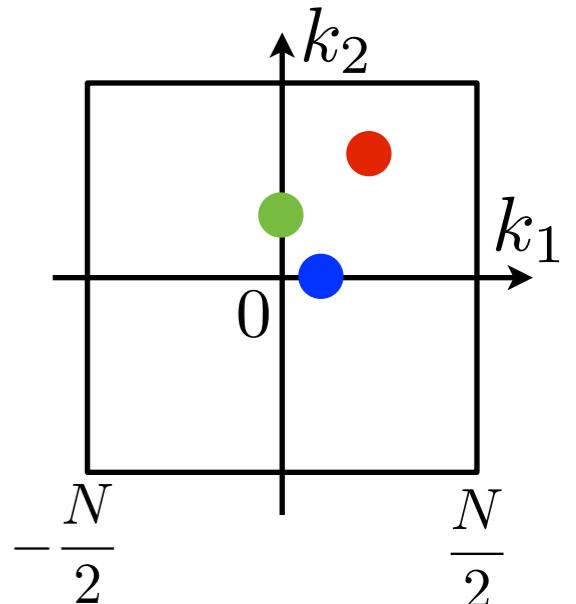
2-D basis:  $e_{k_1, k_2}[n_1, n_2] \stackrel{\text{def.}}{=} e^{\frac{2i\pi}{N} \langle k, n \rangle} = e_{k_1}[n_1]e_{k_2}[n_2]$

Frequency  $(k_1, k_2) \in \{0, \dots, N - 1\}^2$

Fourier transform:  $f \rightarrow \hat{f}$

$$\hat{f}[k_1, k_2] \stackrel{\text{def.}}{=} \langle f, e_{k_1, k_2} \rangle = \sum_{n_1, n_2} f[n_1, n_2] e_{k_1, k_2}[n_1, n_2]$$

Fast Fourier Transform:  $O(N^2 \log(N))$  operations



## Fourier transforms

$$\hat{f}_0(\omega) = \int_{-\infty}^{+\infty} f_0(t) e^{-i\omega t} dt$$

$$\hat{f}_0[m] = \int_0^1 f_0(t) e^{-2i\pi m t} dt$$

$$\hat{f}(\omega) = \sum_{n \in \mathbb{Z}} f[n] e^{i\omega n}$$

$$\hat{f}[m] = \sum_{n=0}^{N-1} f[n] e^{-\frac{2i\pi}{N} mn}$$

Periodization  $f_0(t) \mapsto \sum_n f_0(t + n)$

$f_0(t), t \in \mathbb{R}$	$f_0(t), t \in [0, 1]$	Continuous
$f[n], n \in \mathbb{Z}$	$f[n], 0 \leq n < N$	Discrete
Infinite	Periodic	

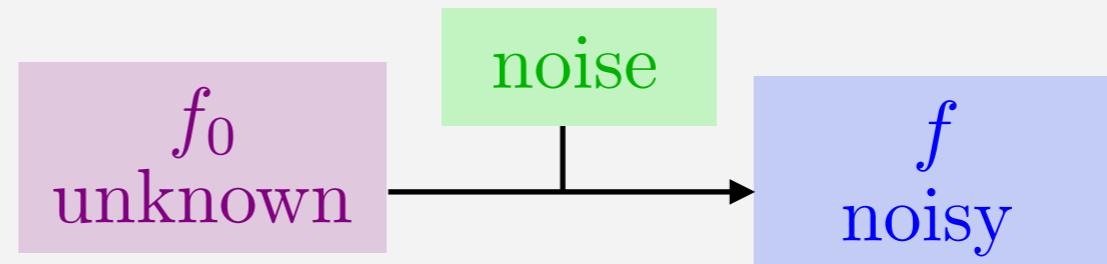
$$\hat{f}(\omega) = \sum_k \hat{f}_0(N(\omega + 2k\pi))$$

Sampling  $\hat{f}_0(\omega) \mapsto \{\hat{f}_0(k)\}_k$

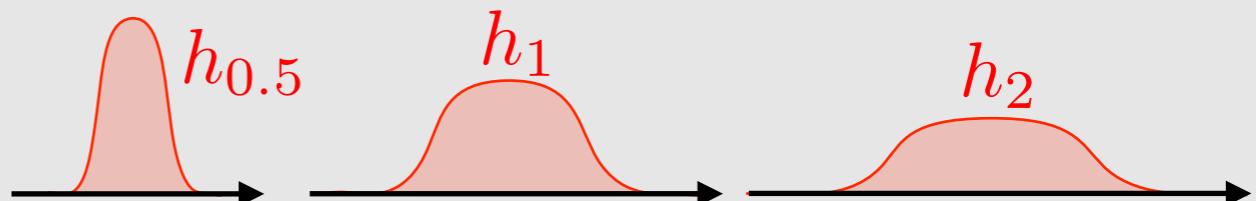
$\hat{f}_0(\omega), \omega \in \mathbb{R}$	$\hat{f}_0[k], k \in \mathbb{Z}$	Infinite
$\hat{f}(\omega), \omega \in [0, 2\pi]$	$\hat{f}[k], 0 \leq k < N$	Periodic
Continuous	Discrete	

Fourier transform  
Isometry  $f \mapsto \hat{f}$

## Data generation model:

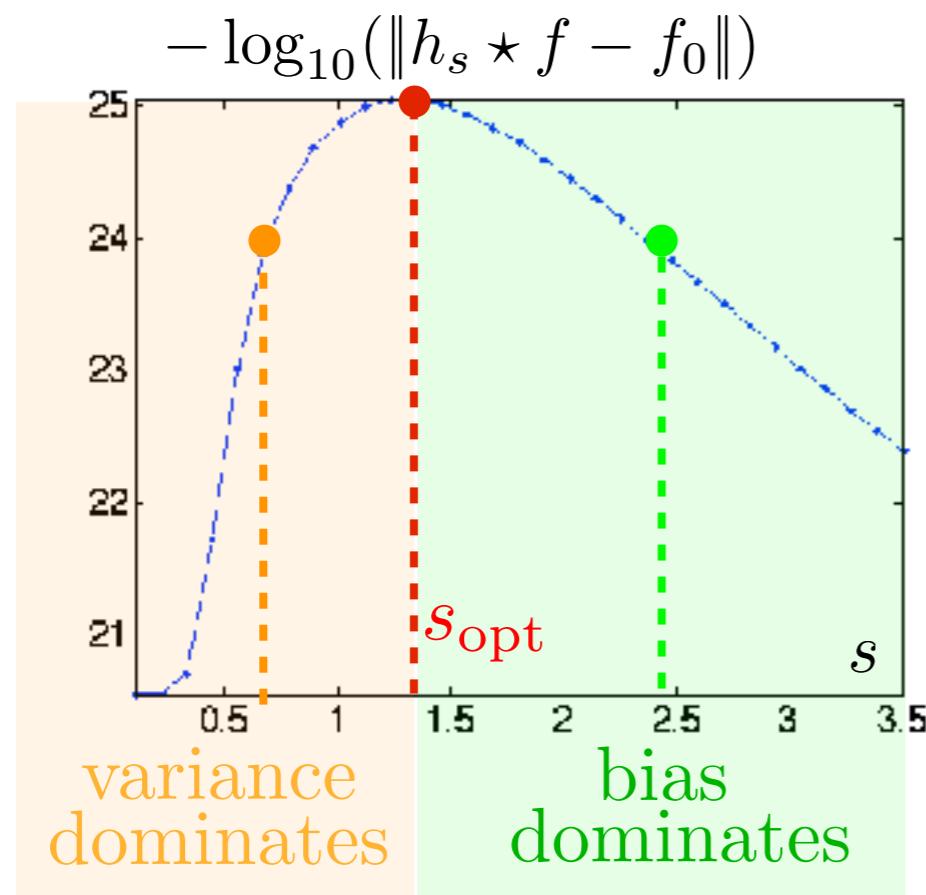


Dilated filter:  $h_s = \frac{1}{s^{d/2}} h(\cdot/s)$



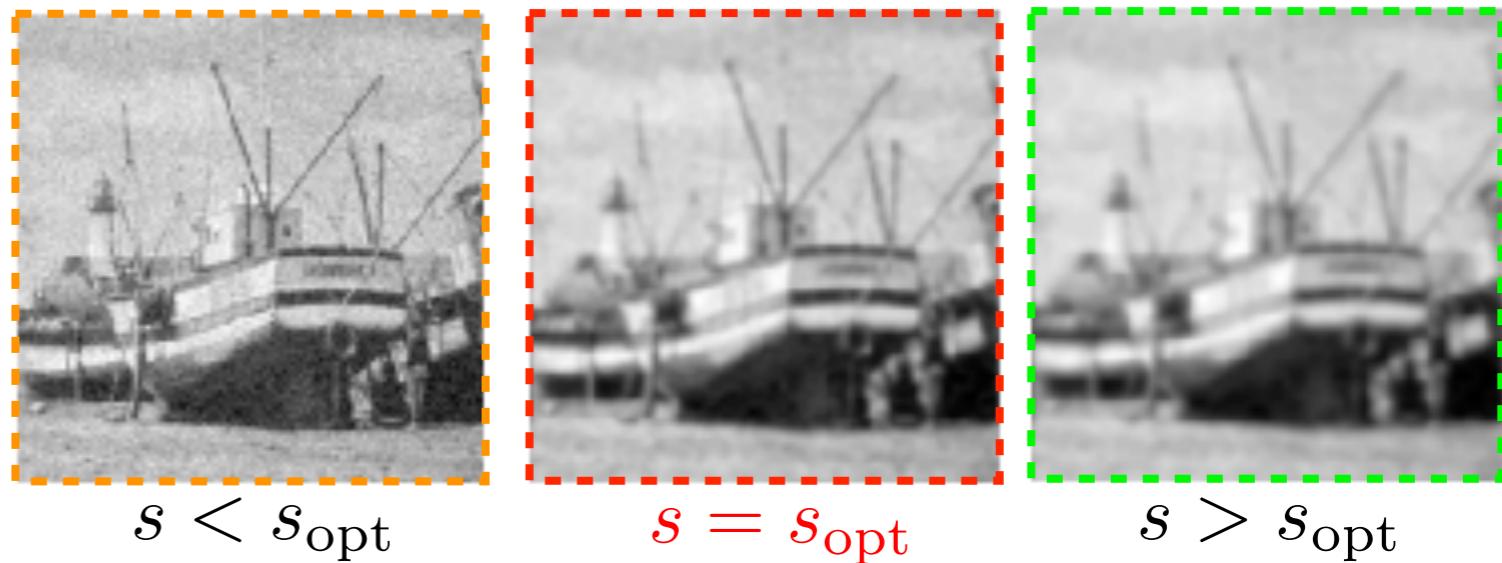
## Linear invariant denoiser:

$$f \xrightarrow{\text{convolution}} f \star h_s \stackrel{\text{def.}}{=} \sum_k f[k] h_s[\cdot - k]$$

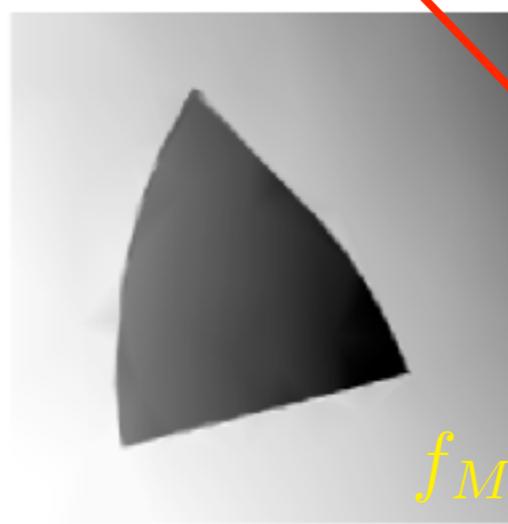
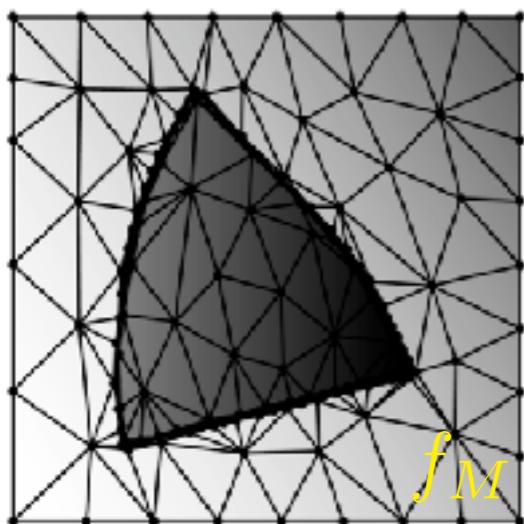
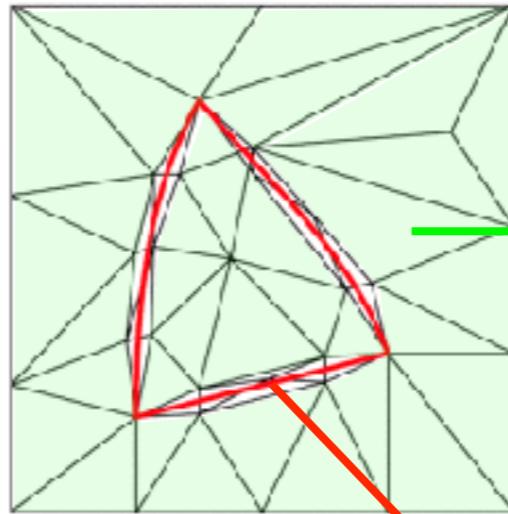
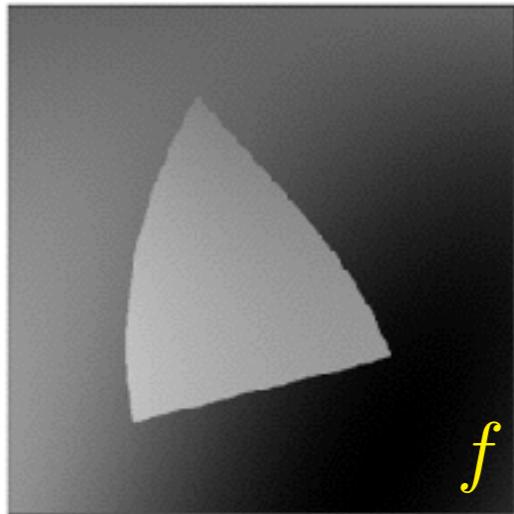


$$\|h_s \star f - f_0\| \leq \|h_s \star (f - f_0)\| + \|h_s \star f_0 - f_0\|$$

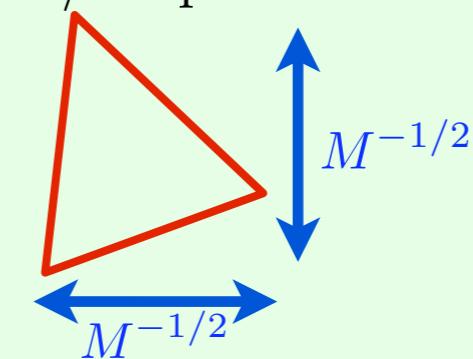
error	variance	bias
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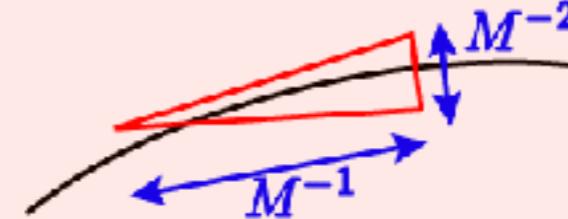
Piecewise linear approximation using  $M$  triangles:



Regular areas:  
 $\sim M/2$  equilateral triangles.



Singular areas:  
 $\sim M/2$  anisotropic triangles.



*Theorem:* If  $f$  is  $C^2$  outside  $C^2$  curves,

$$\|f - f_M\|^2 = O(M^{-2})$$

→ to be compared with wavelets:

$$\|f - f_M\|^2 = O(M^{-1})$$



J-M Mirebeau



Thin plate spline:  $\min_f \sum_{k=1}^K \|y_k - f(x_k)\|^2 + \lambda \|\Delta f\|^2$

Theorem:  $\exists A \in R^{2 \times 2}, b \in \mathbb{R}^2, (w_k \in \mathbb{R}^2)_k$

$$f(x) = Ax + b + \sum_{k=1} w_k \varphi(\|x - x_k\|) \quad \varphi(r) = r^2 \log(r)$$

solve a linear system

