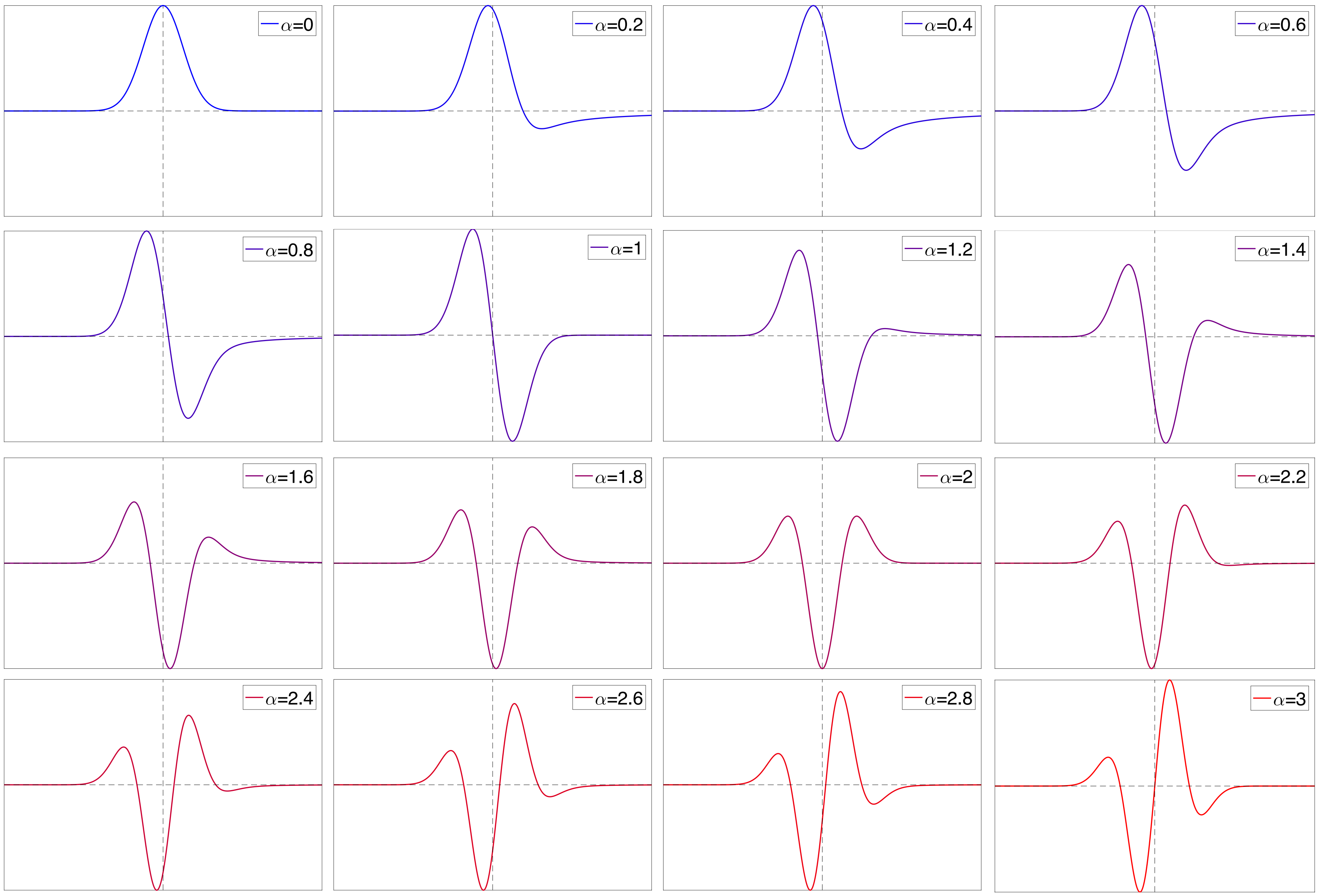
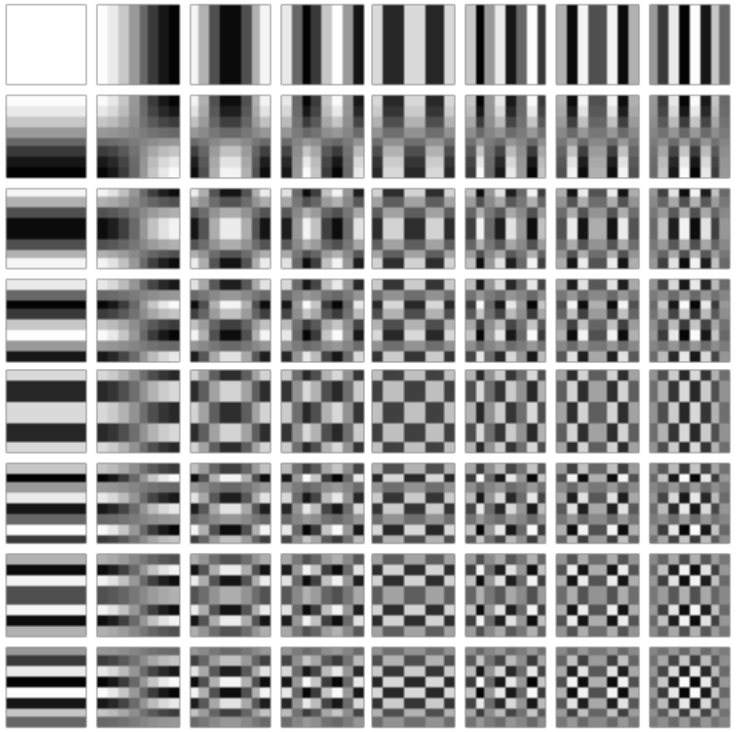
Signal and Image Processing

Fourier transform: $\mathcal{F}(f)(\omega) \stackrel{\text{def.}}{=} \int_{\mathbb{R}} f(x) e^{-\mathrm{i}\omega x} \mathrm{d}x$

Fractional derivative: $\mathcal{F}(f^{(\alpha)}) \stackrel{\text{def.}}{=} (\mathrm{i}\omega)^{\alpha} \mathcal{F}(f)(\omega)$



DCT-1:	$\cos jk\frac{\pi}{N-1}$	(divide by $\sqrt{2}$ when j or k is 0 or $N-1$)
DCT-2:	$\cos\left(j+\frac{1}{2}\right)k\frac{\pi}{N}$	(divide by $\sqrt{2}$ when $k=0$)
DCT-3:	$\cos j\left(k+\frac{1}{2}\right)\frac{\pi}{N}$	(divide by $\sqrt{2}$ when $j=0$)
DCT-4:	$\cos\left(j+\frac{1}{2}\right)\left(k+\frac{1}{2}\right)\frac{\pi}{N}$	

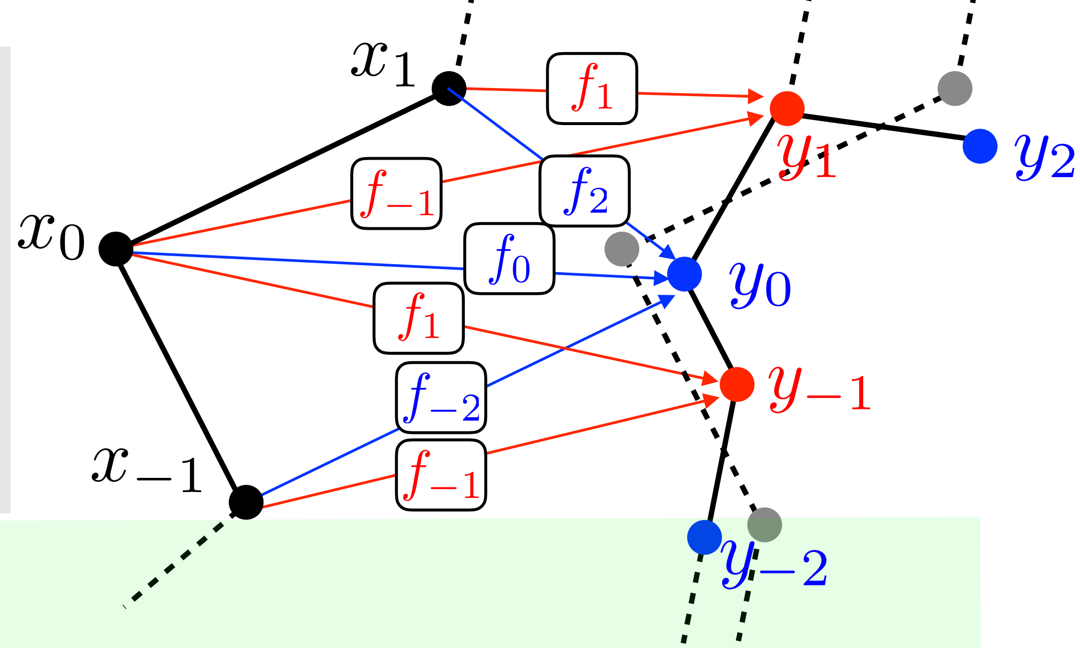


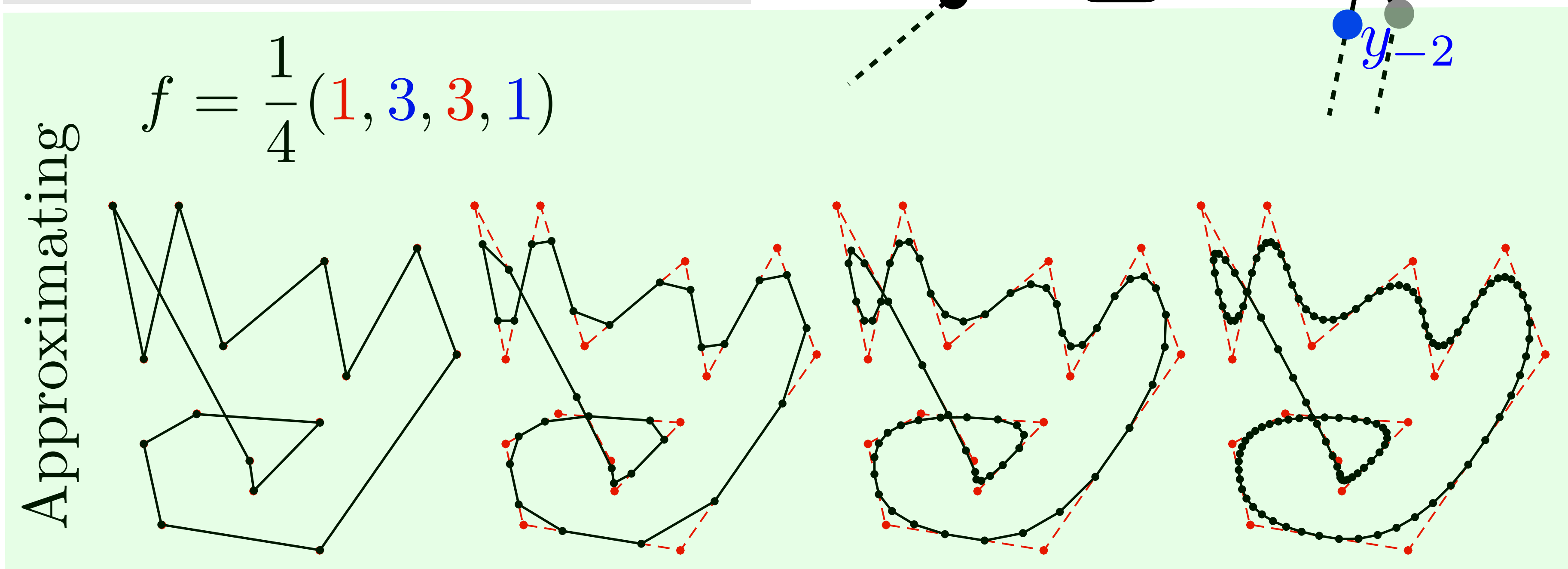
The discrete case has a new level of variety and complexity, often appearing in the boundary conditions [G. Strang - SIAM review, 1999]

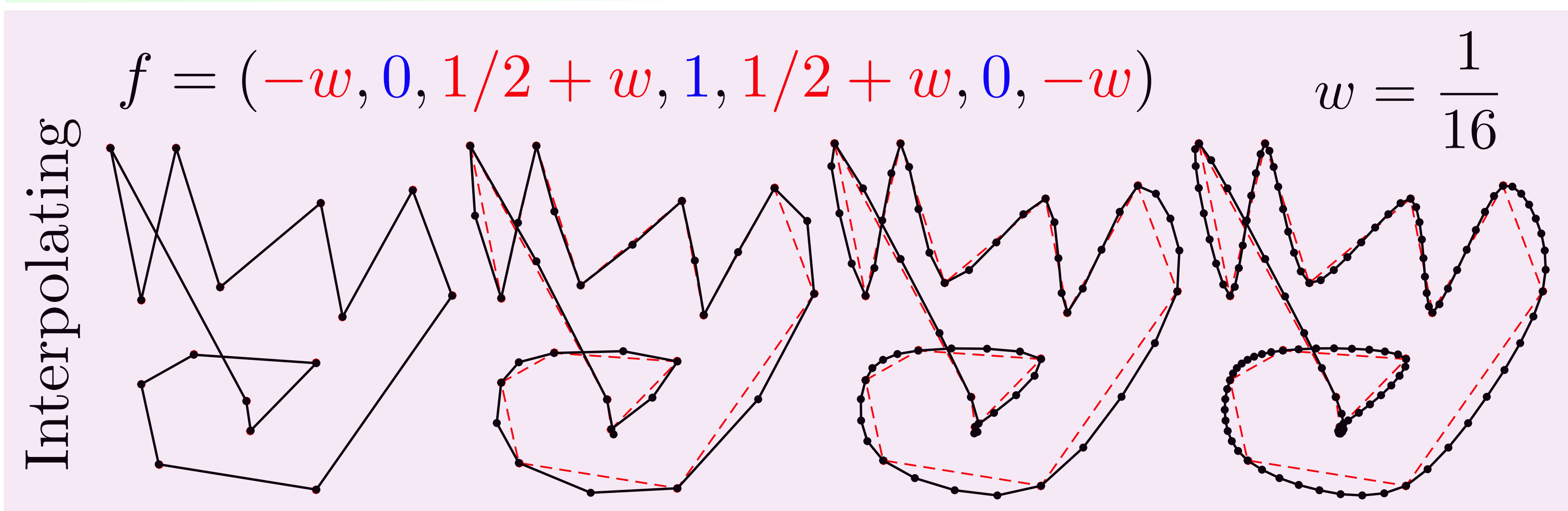
Refinement scheme:

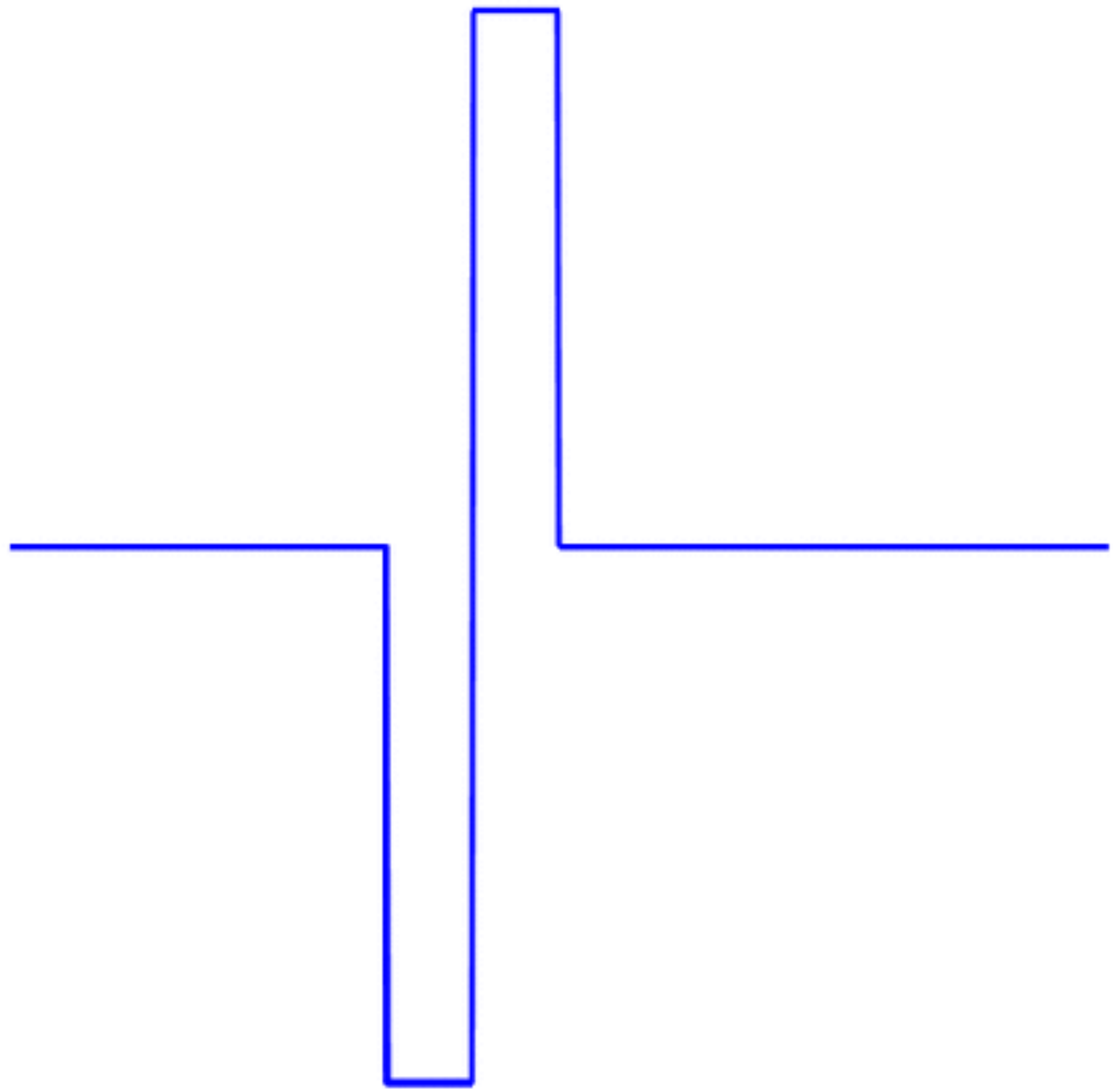
$$y_{2k} = \sum_i f_{2i} x_{k+i}$$

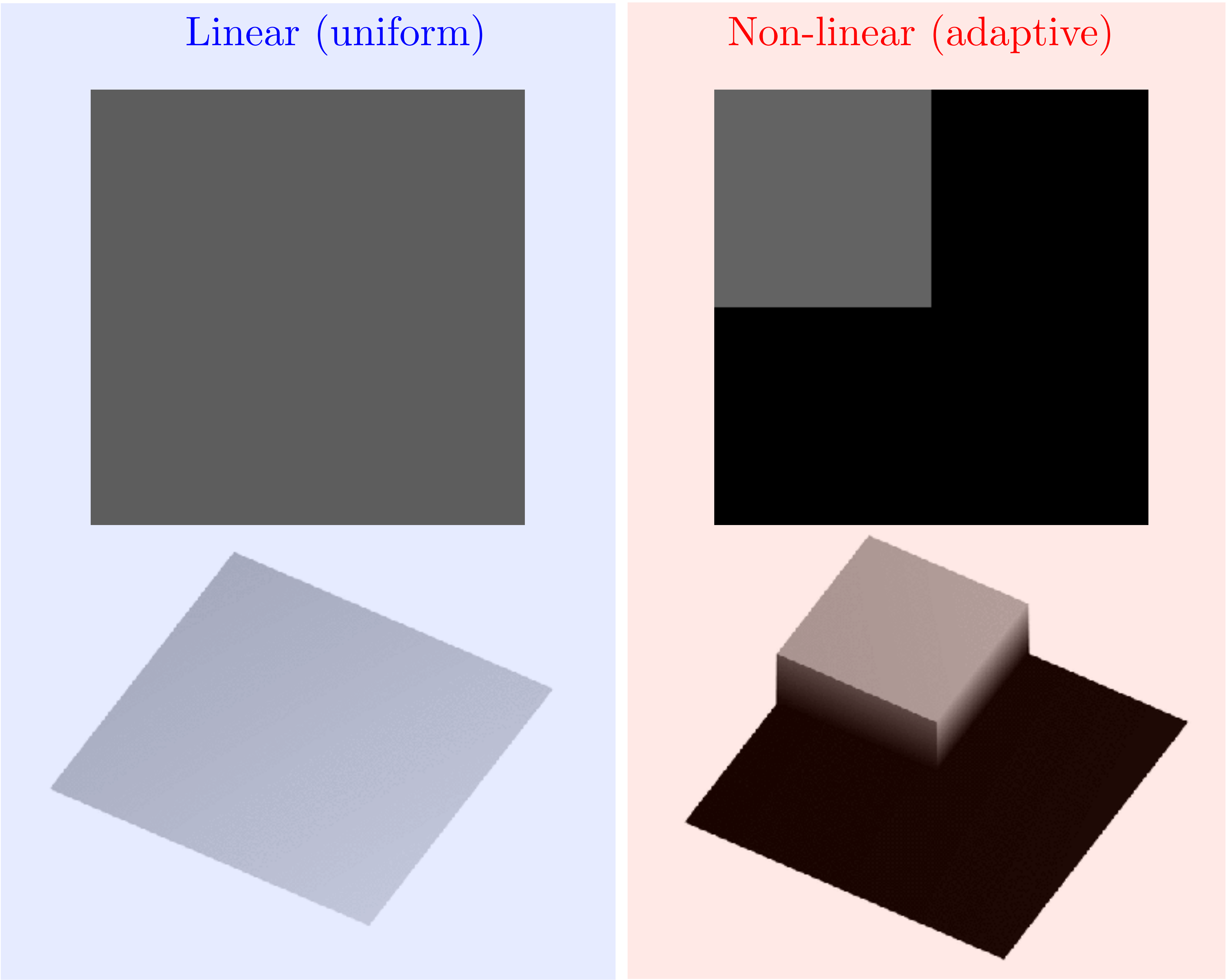
$$y_{2k+1} = \sum_i f_{2i+1} x_{k+i}$$



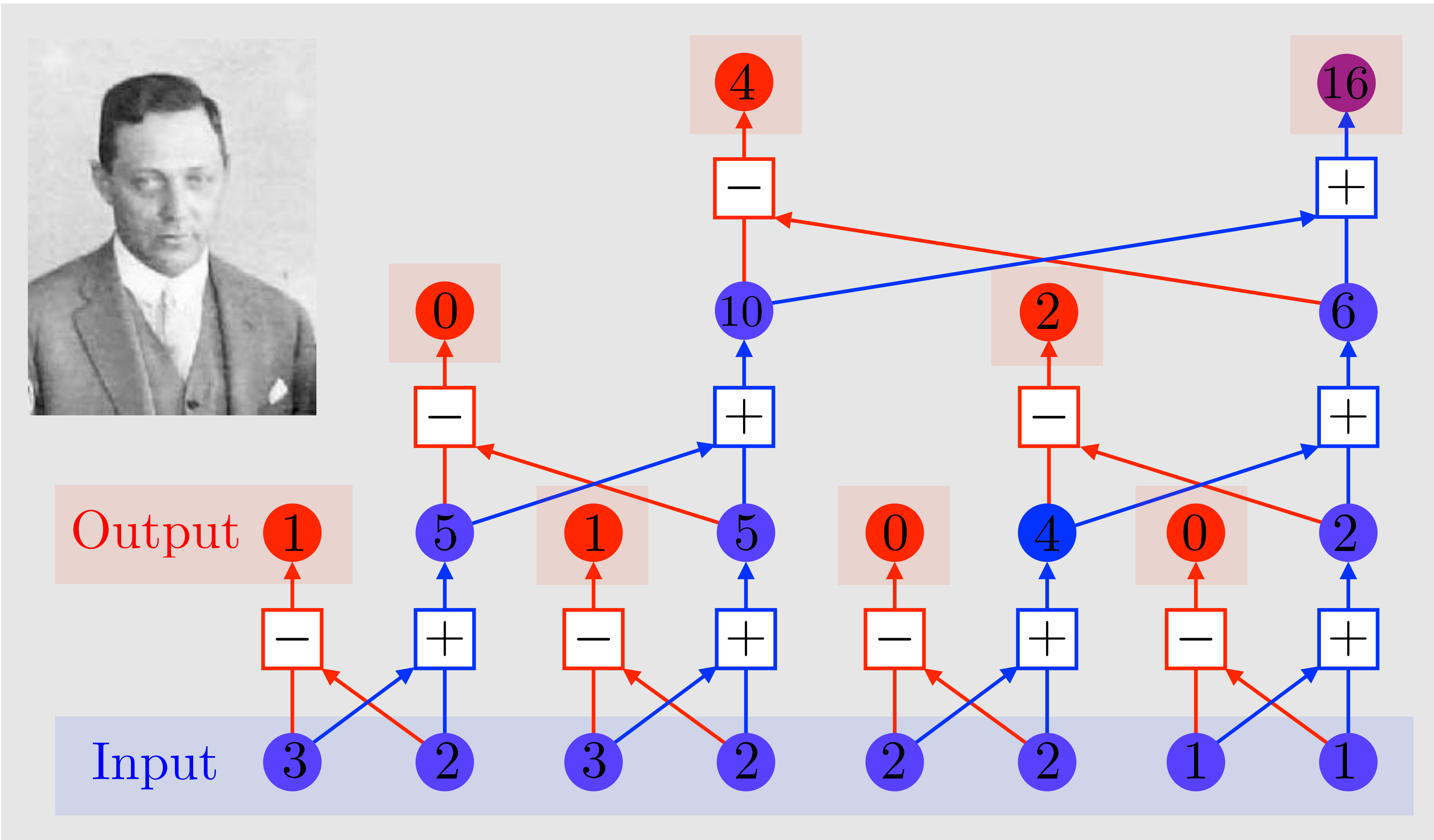







Linear (uniform)
Non-linear (adaptive)

$$(3, 2, 3, 2, 2, 2, 1, 1) \xrightarrow[\text{transform}]{\text{Haar}} (1, 1, 0, 0, 0, 2, 4, 16)$$

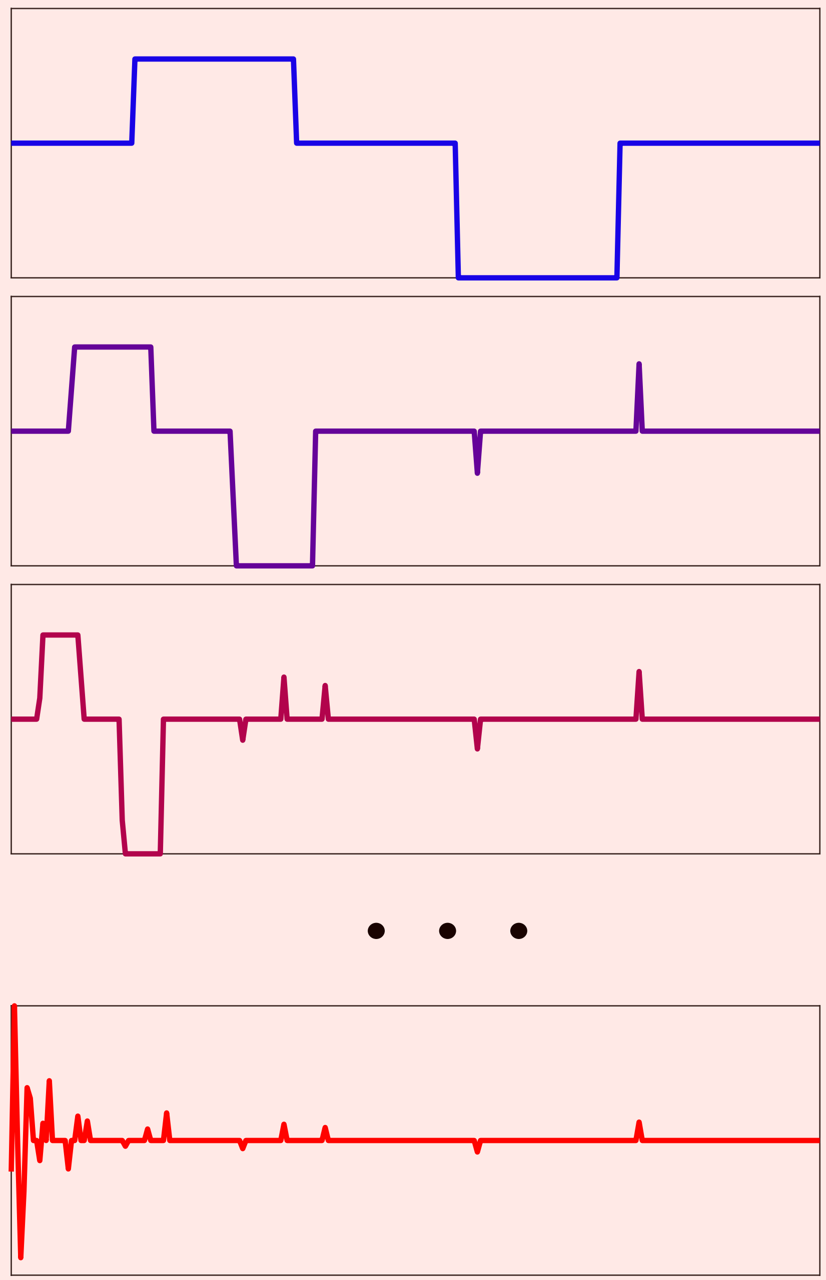


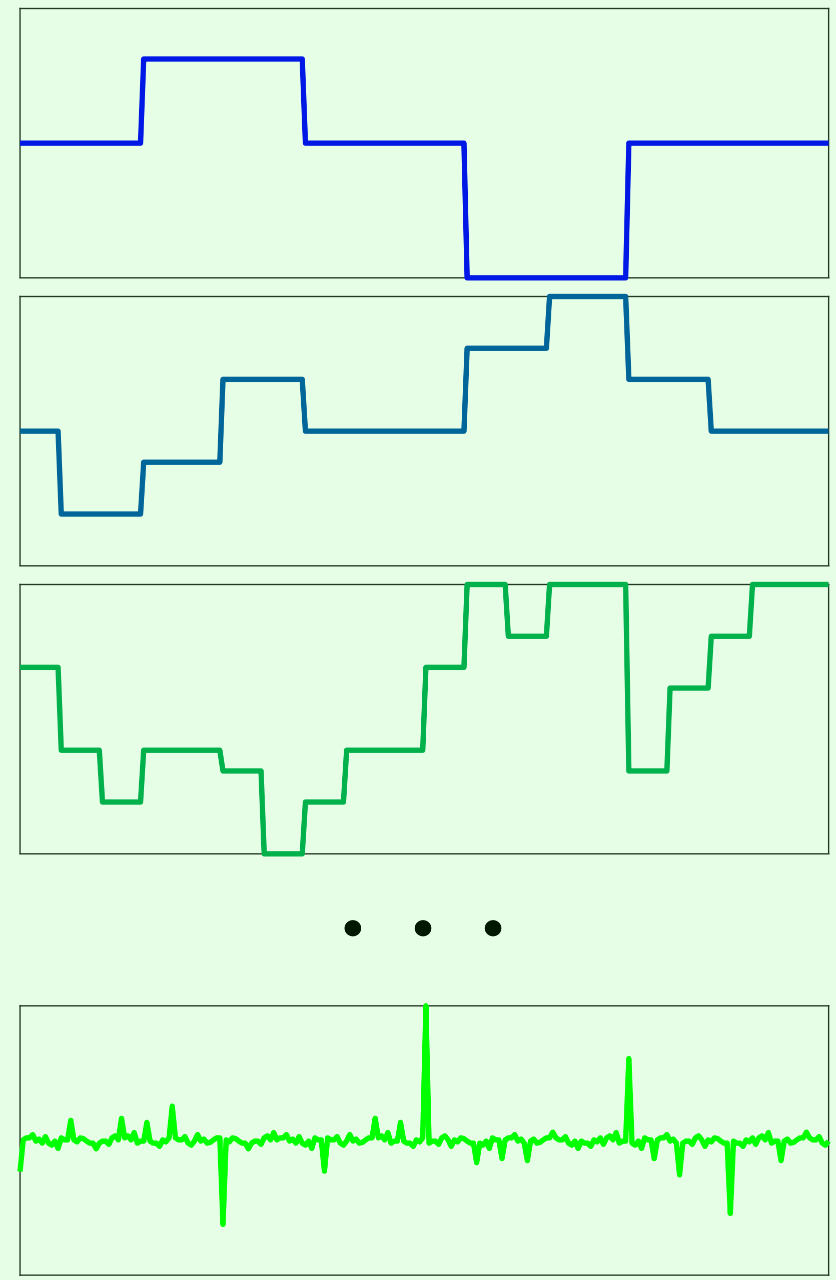
Walsh: $n\log(n)$ operations.

$$W(f) = (W(f_{1:\frac{n}{2}}) + W(f_{\frac{n}{2}+1:n}), W(f_{1:\frac{n}{2}}) - W(f_{\frac{n}{2}+1:n}))$$

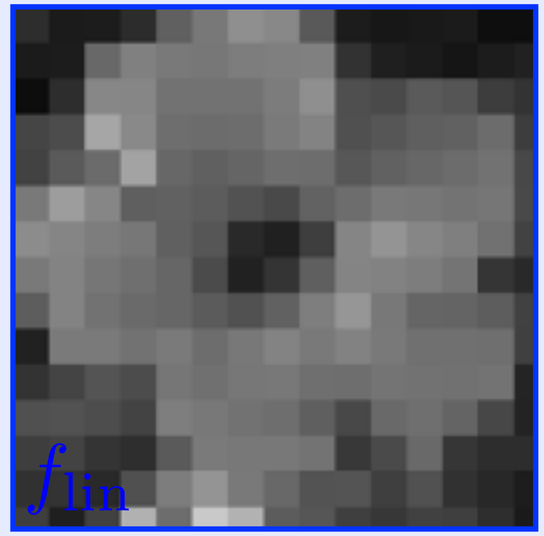
Haar: $2n$ operations.

$$H(f) = (H(f_{1:2:n-1} + f_{2:2:n}), f_{1:2:n-1} - f_{2:2:n})$$



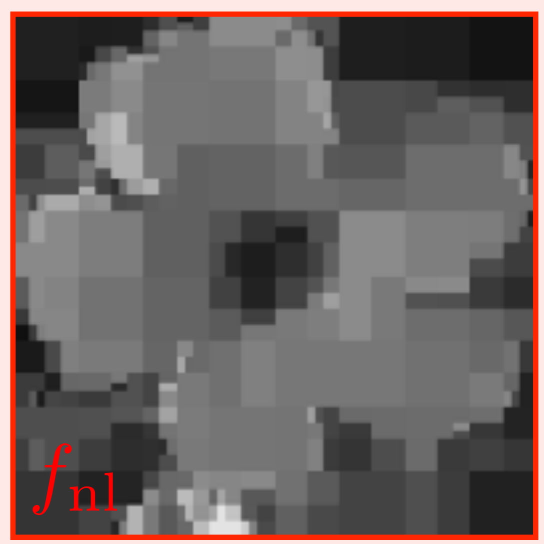


f



Orthonormal basis $(\psi_m)_m$.

$$f_{\text{lin}} = \sum_{m=1}^{n} \langle f,\, \psi_m \rangle \psi_m$$



$$f_{\text{nl}} = \sum_{i=1}^{n} \langle f,\, \psi_{m_i} \rangle \psi_{m_i}$$

$$|\langle f,\, \psi_{m_1} \rangle| \geqslant |\langle f,\, \psi_{m_2} \rangle| \geqslant \ldots$$

Alphabet K, probabilities $p = (p_k)_{k \in K}$.

Binary code: $k \in K \mapsto c_k \in \{0,1\}^{|c_k|}$.

Average code length: $L(c) = \sum_k p_k |c_k|$

T = $\texttt{Huffman}(K,p)$

– If $|K| = 1$, $T = \{\cdot\}$.

– If $|K| > 1$, sort $p_a \leqslant p_b \leqslant \ldots$.

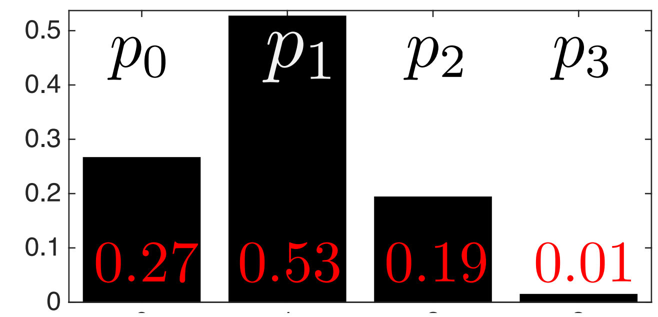
$$K' \overset{\text{def.}}{=} (K \backslash \{a,b\}) \cup \{z\}$$

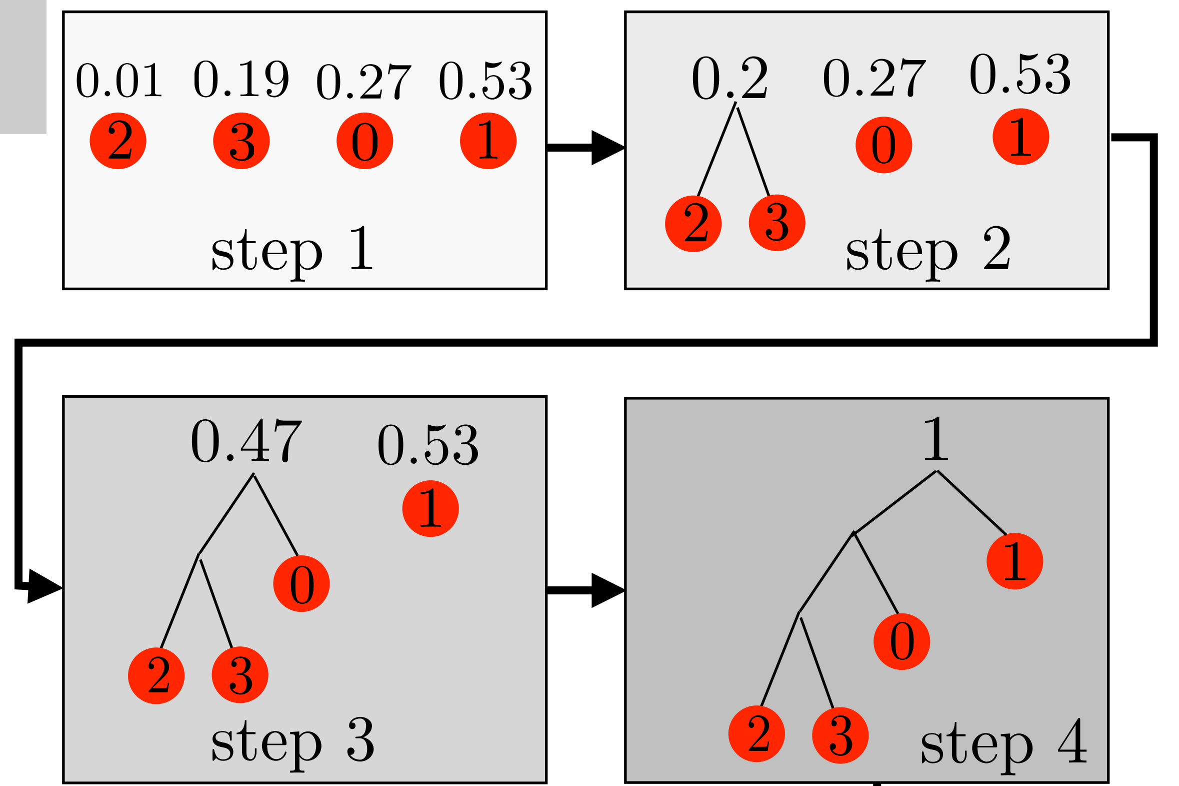
$$p'_k = \begin{cases} p_a + p_b \text{ si } k = z, \\ p_k \text{ otherrwise.} \end{cases}$$

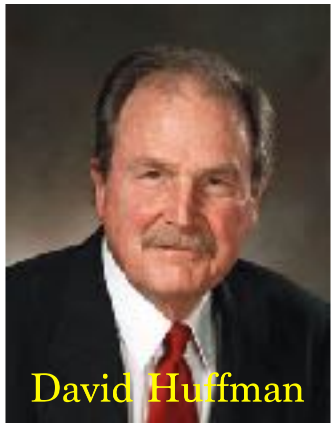
$$T' \overset{\text{def.}}{=} \texttt{Huffman}(K',p')$$

T: add (a,b) under z.

Theorem: let $c_{\texttt{H}}$ the code of $\texttt{Huffman}(K,p)$.
For any prefix code c, $L(c) \geqslant L(c_{\texttt{H}})$







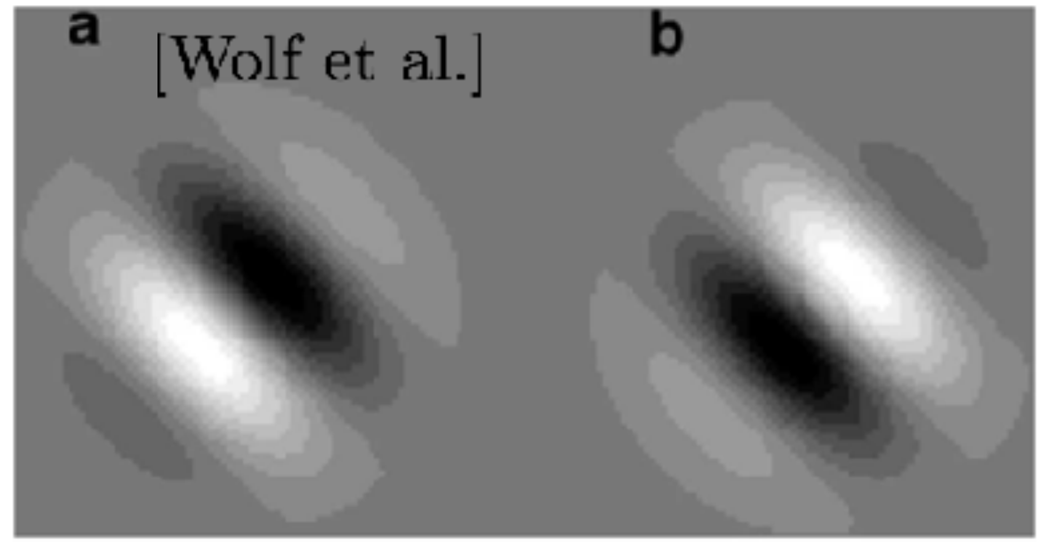
Continuous

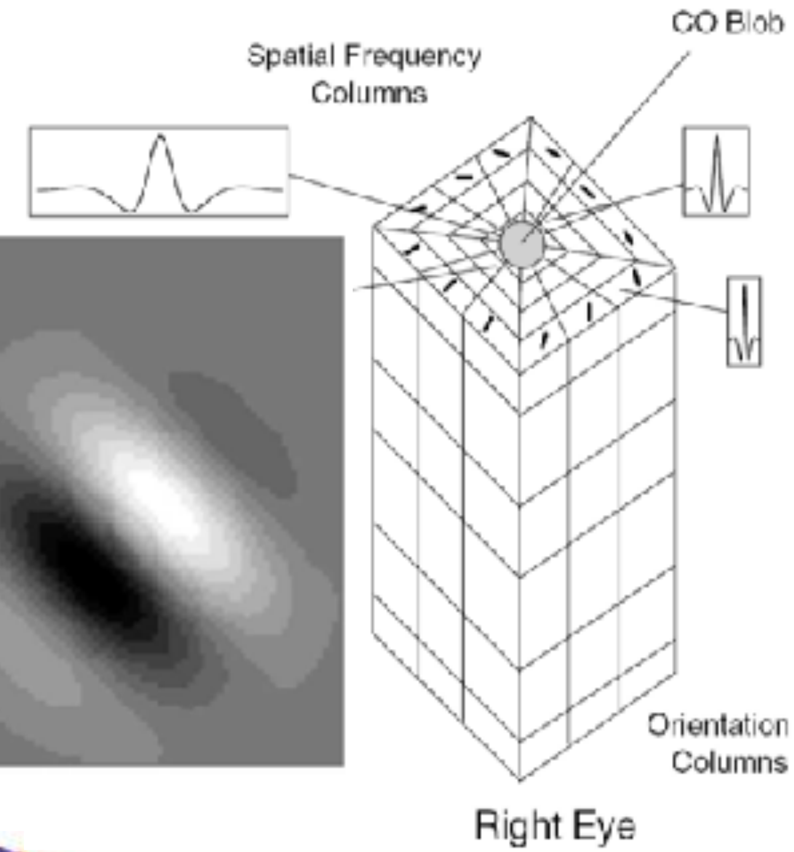
Acquisition
$\longrightarrow$ cones of the retina

Discrete

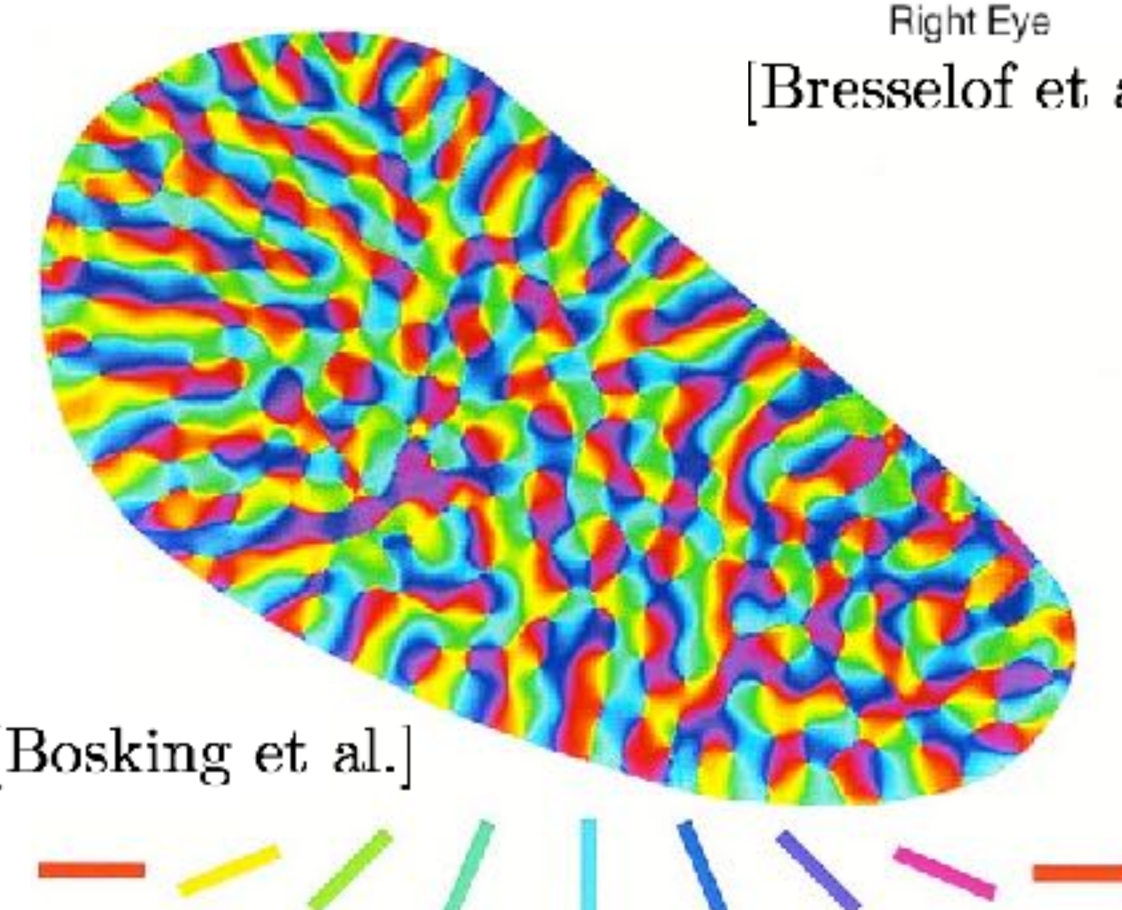
Wavelet transform
$\longrightarrow$ simple cells in V1

Multiscale





[Bresselof et al.]



[Bosking et al.]

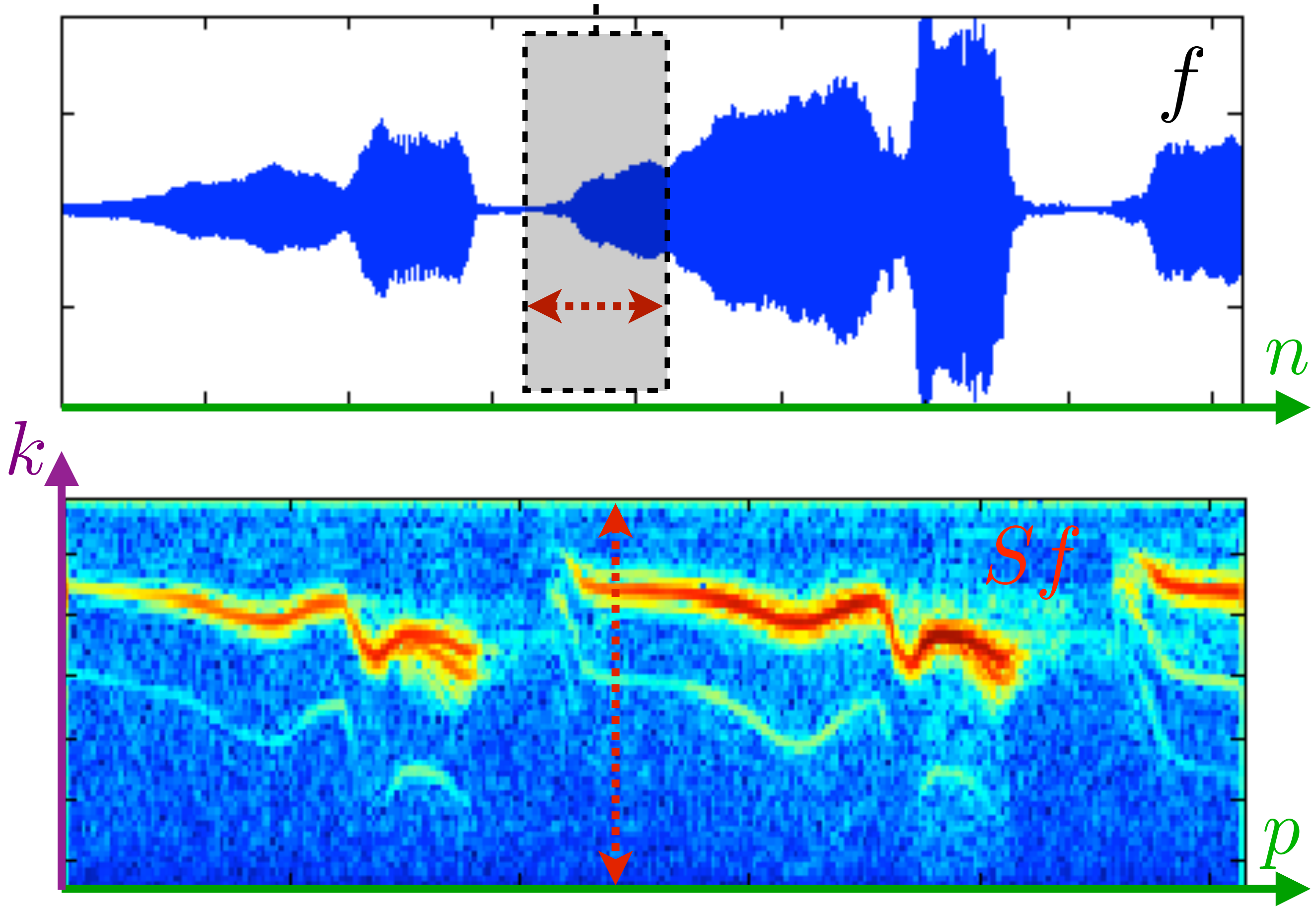
Spectrogramm / Short-time Fourier transform:

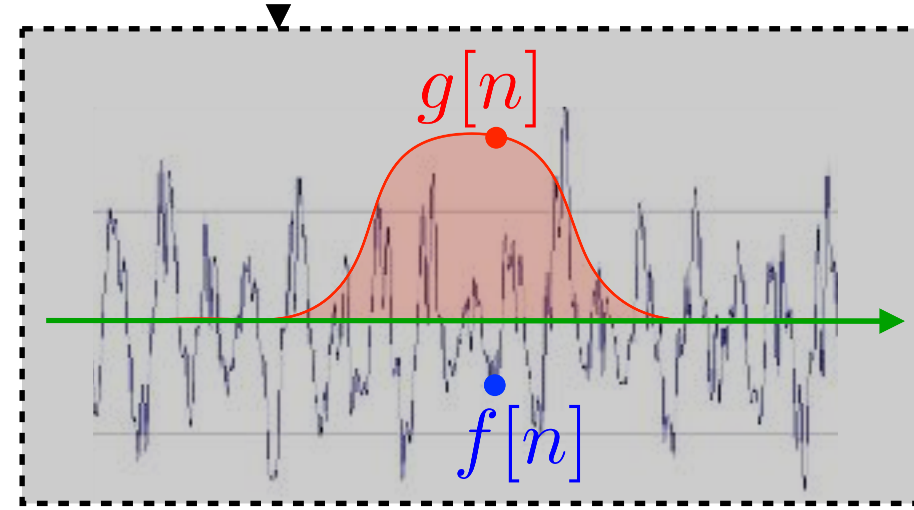
$$Sf[k,p] = Q^{-1/2} \sum_n f[n] g[\Delta_x p - n] e^{-\frac{2i\pi}{Q} kn}$$

frequency (k) time (p)

signal f

window g





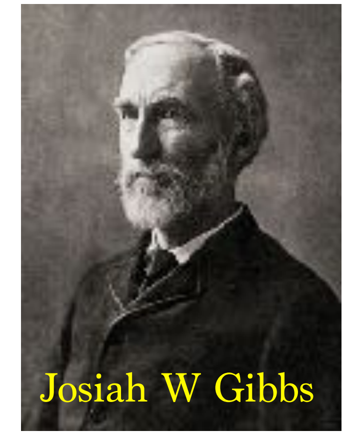
Computation:

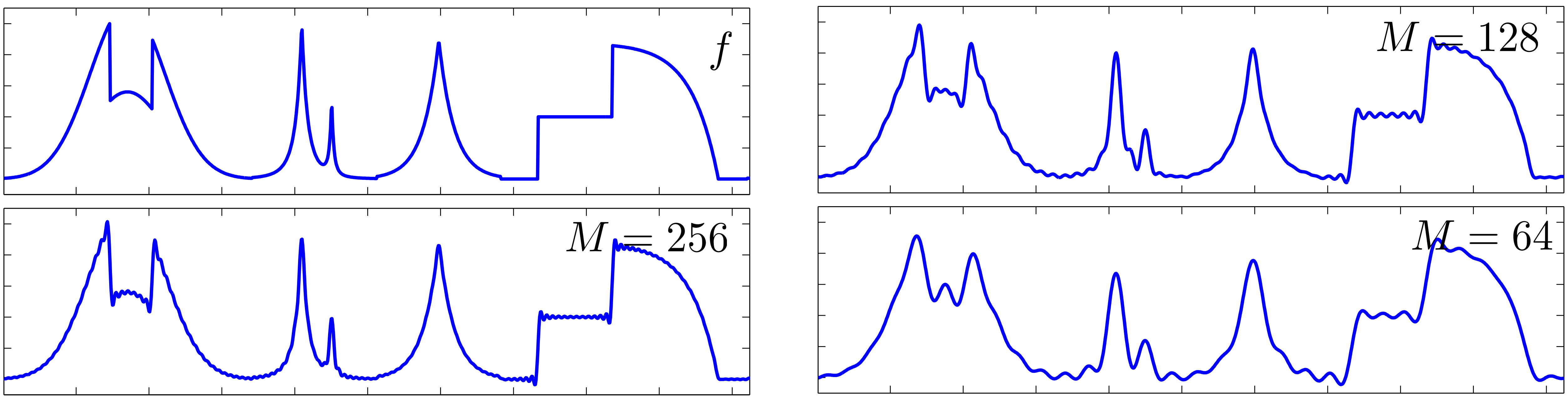
N FFTs of size Q

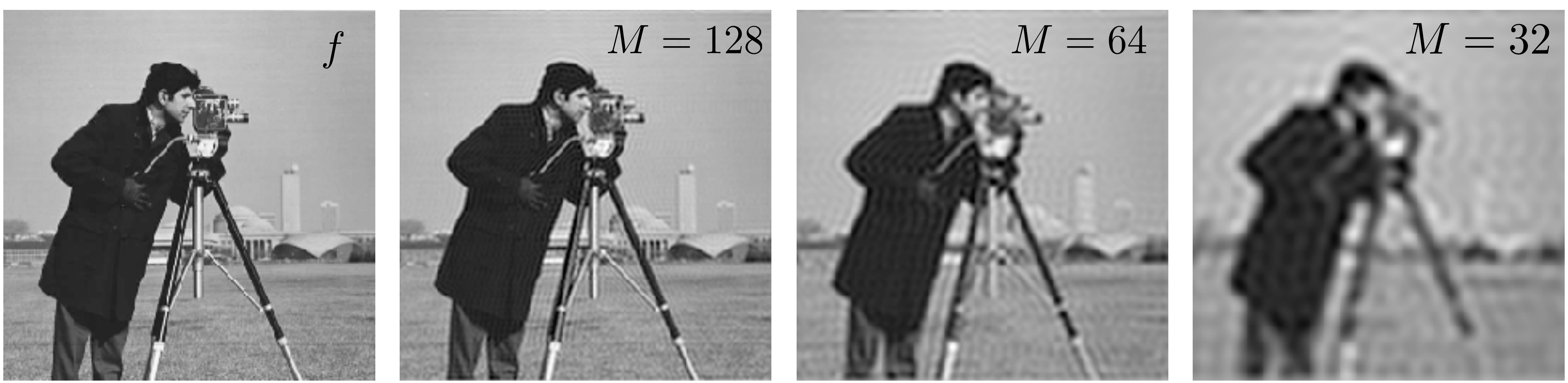
$O(NQ\log(Q))$ operations.

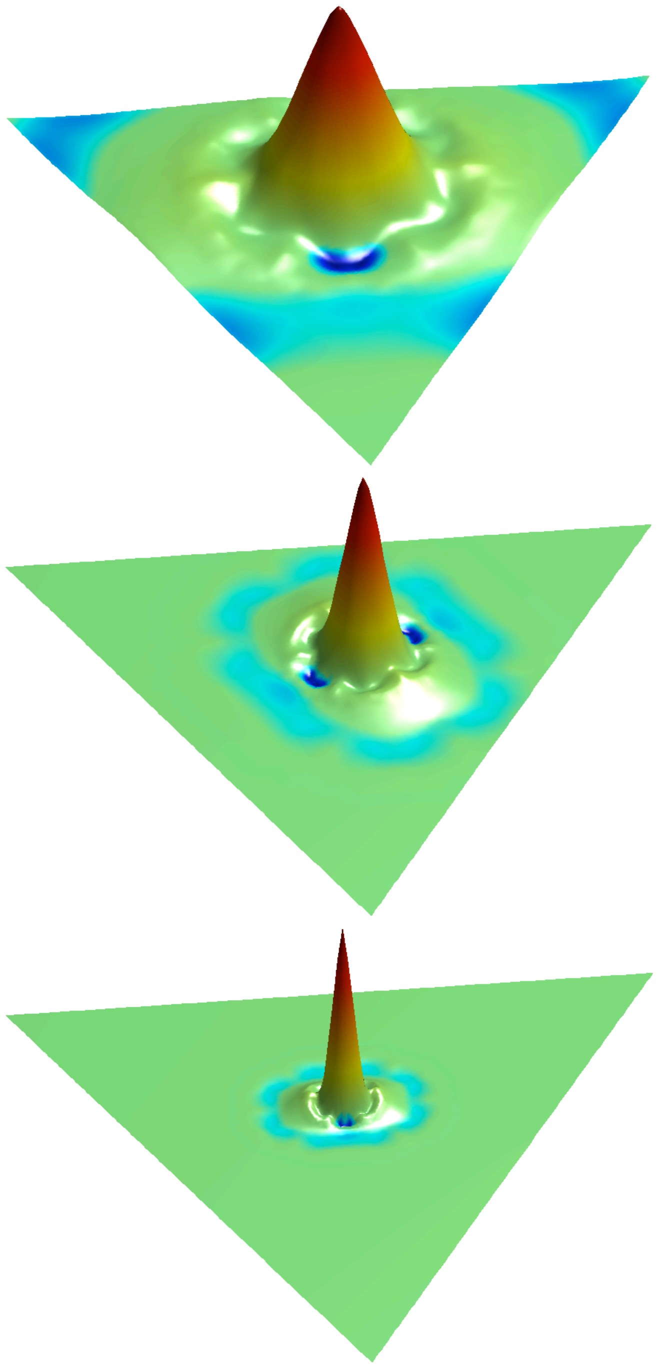
Fourier atoms: $e_m \overset{\text{def.}}{=} e^{2\mathrm{i}\pi\langle m,\, x\rangle}$ Frequency m.

Linear Fourier approximation: $f_M \overset{\text{def.}}{=} \sum_{|m|\leqslant M/2} \langle f,\, e_m\rangle e_m$

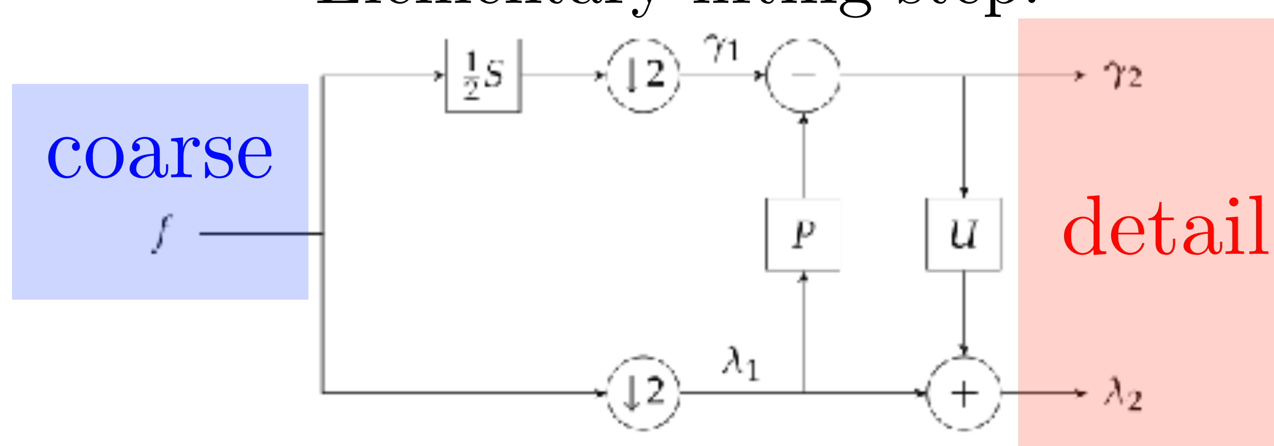


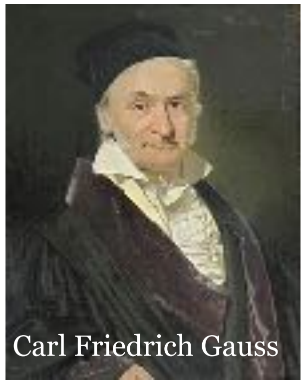






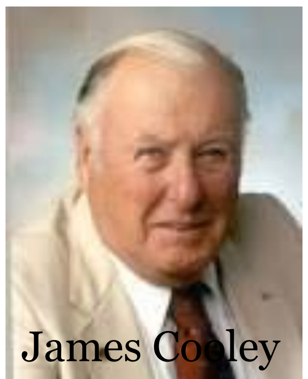
Elementary lifting step:

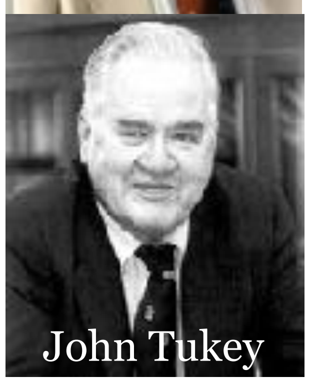




Discrete Fourier transform: $\mathcal{F}_n(f)_\ell \stackrel{\text{def.}}{=} \sum_{k=0}^{n-1} f_k e^{\frac{2i\pi}{n} k\ell}$

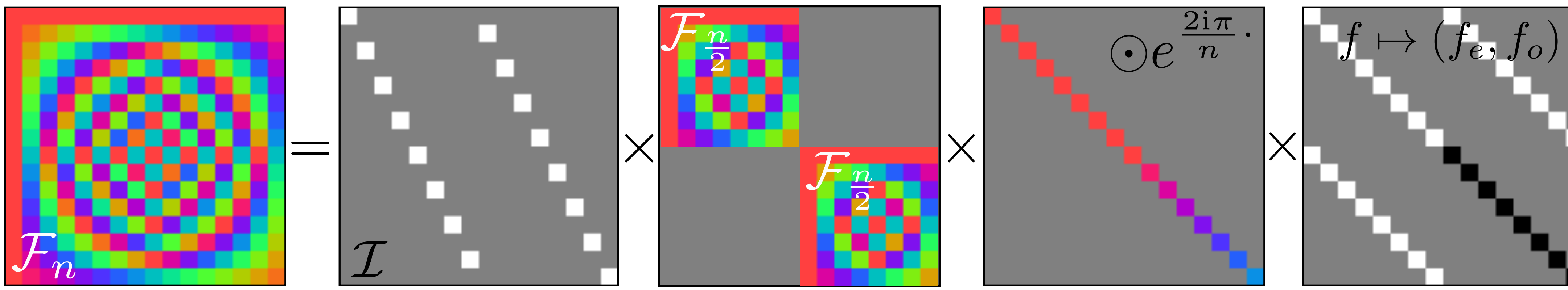
...





FFT (1 step): $\mathcal{F}_n(f) = \mathcal{I}(\mathcal{F}_{\frac{n}{2}}(f^+), \mathcal{F}_{\frac{n}{2}}(f^- \odot e^{\frac{2i\pi}{n}\cdot}))$

$f^\pm \stackrel{\text{def.}}{=} f_\cdot \pm f_{\cdot + \frac{n}{2}}$

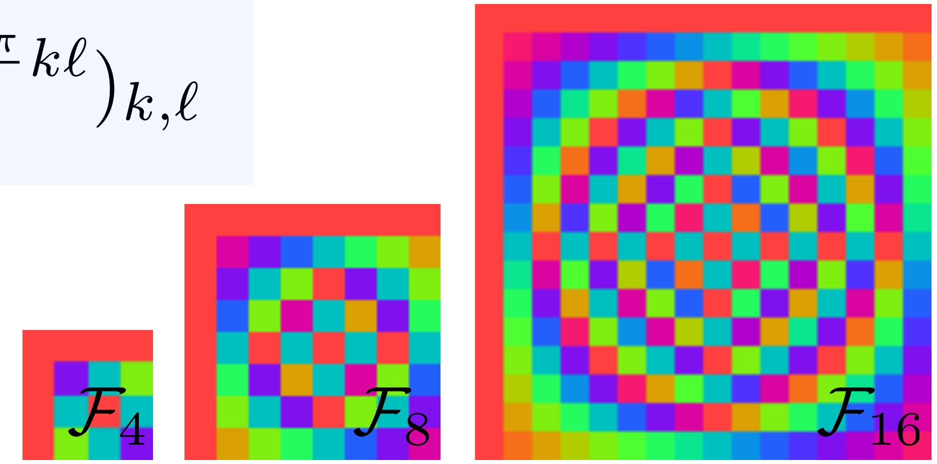


$\text{Time}(\mathcal{F}_n) = 2\text{Time}(\mathcal{F}_{\frac{n}{2}}) + cn \implies \text{Time}(\mathcal{F}_n) = cn\log_2(n)$

Discrete Fourier matrix: $\mathcal{F}_n \overset{\text{def.}}{=} (e^{\frac{-2\mathrm{i}\pi}{n} k\ell})_{k,\ell}$

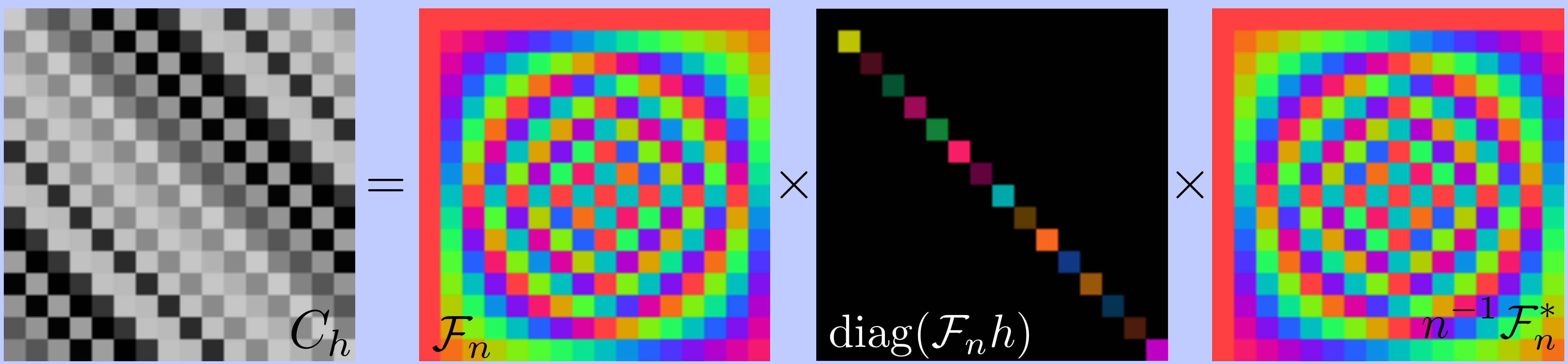
Convolution:

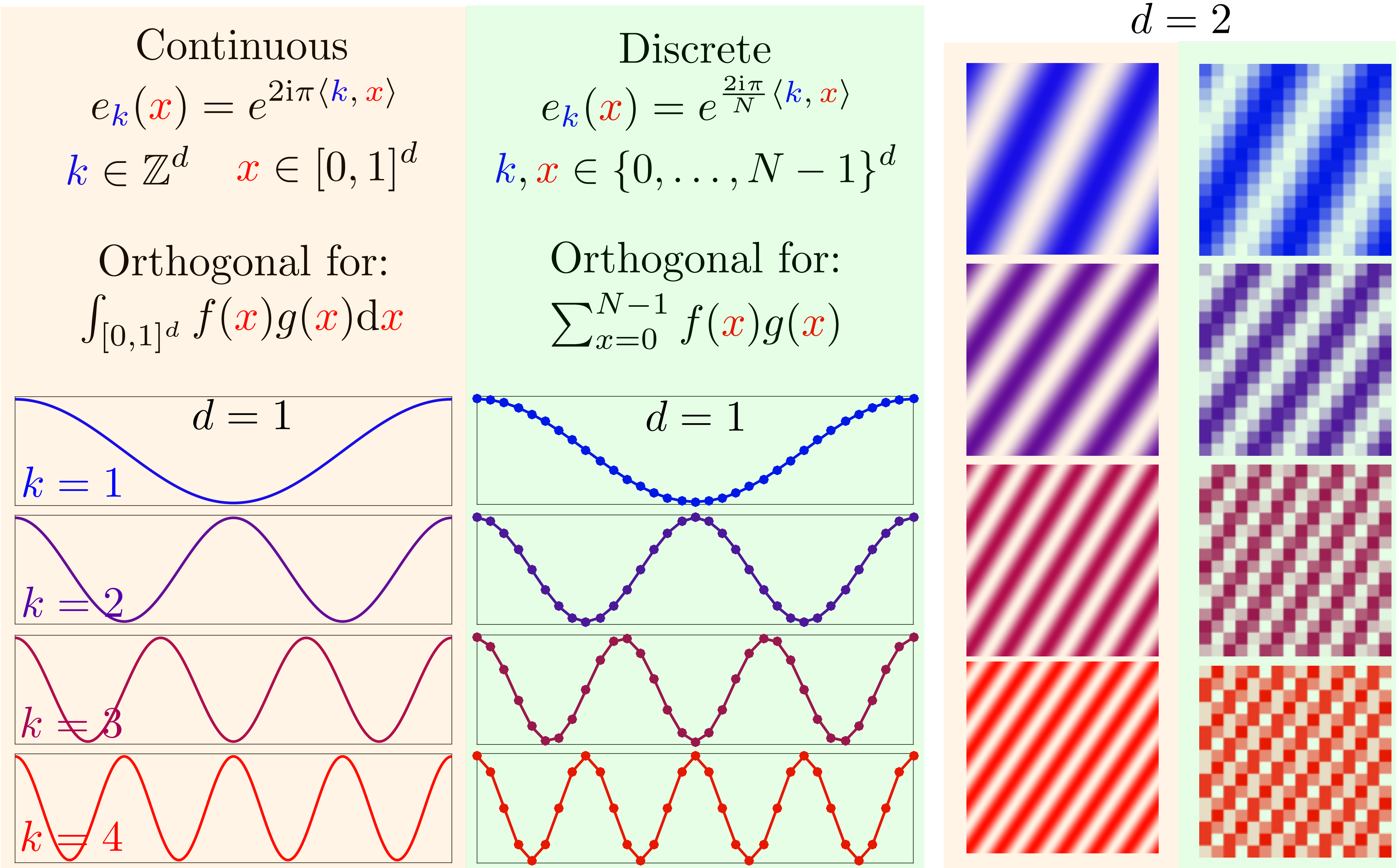
$$(f \star h)_i \overset{\text{def.}}{=} \sum_j f(j)h(i-j)$$



Convolution matrix: $C_h = (h(i-j \bmod n))_{i,j}$ i.e. $C_h(f) = f \star h$

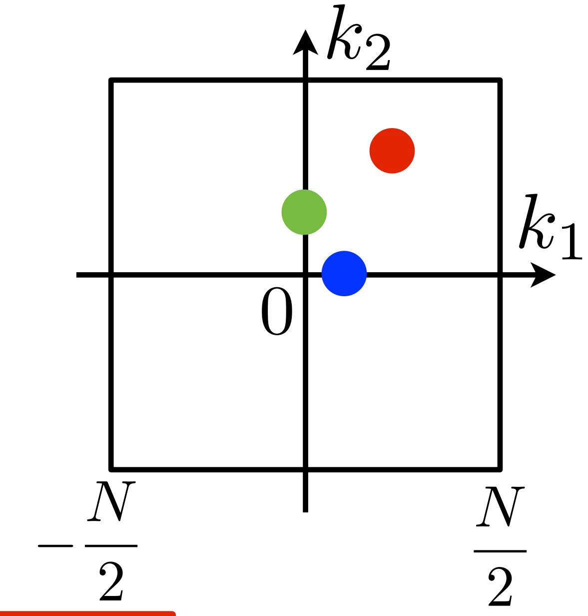
Fourier convolution theorem: $\mathcal{F}_n(f \star h) = \mathcal{F}_n(f) \odot \mathcal{F}_n(h)$





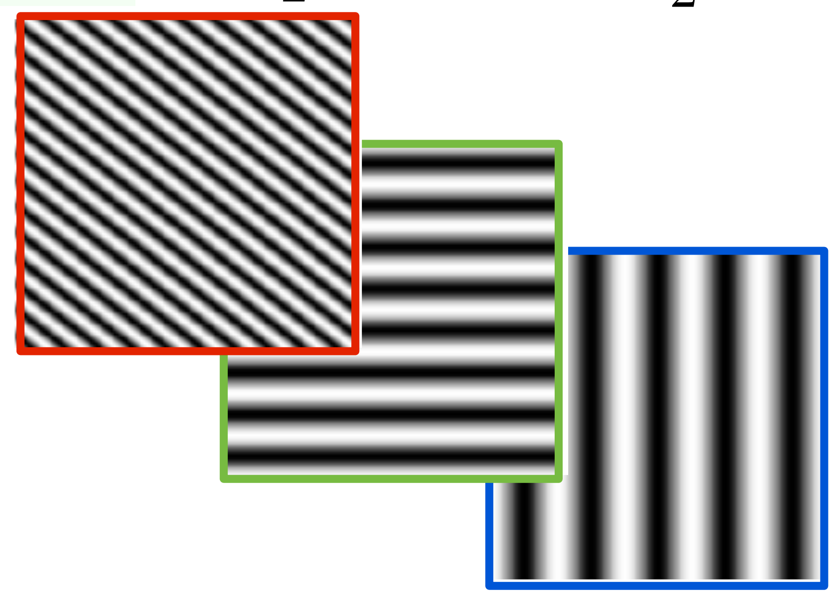
Continuous
$e_k(x) = e^{2i\pi\langle k, x\rangle}$
$k \in \mathbb{Z}^d \quad x \in [0,1]^d$
Orthogonal for:
$\int_{[0,1]^d} f(x)g(x)\mathrm{d}x$
Discrete
$e_k(x) = e^{\frac{2i\pi}{N}\langle k, x\rangle}$
$k, x \in \{0, \ldots, N-1\}^d$
Orthogonal for:
$\sum_{x=0}^{N-1} f(x)g(x)$
$d = 1$
$d = 1$
$k = 1$
$k = 2$
$k = 3$
$k = 4$
$d = 2$

1-D discrete Fourier basis: $e_k[n] \overset{\text{def.}}{=} e^{\frac{2i\pi}{N} kn}$



2-D basis: $e_{k_1,k_2}[n_1,n_2] \overset{\text{def.}}{=} e^{\frac{2i\pi}{N} \langle k,\, n \rangle} = e_{k_1}[n_1] e_{k_2}[n_2]$

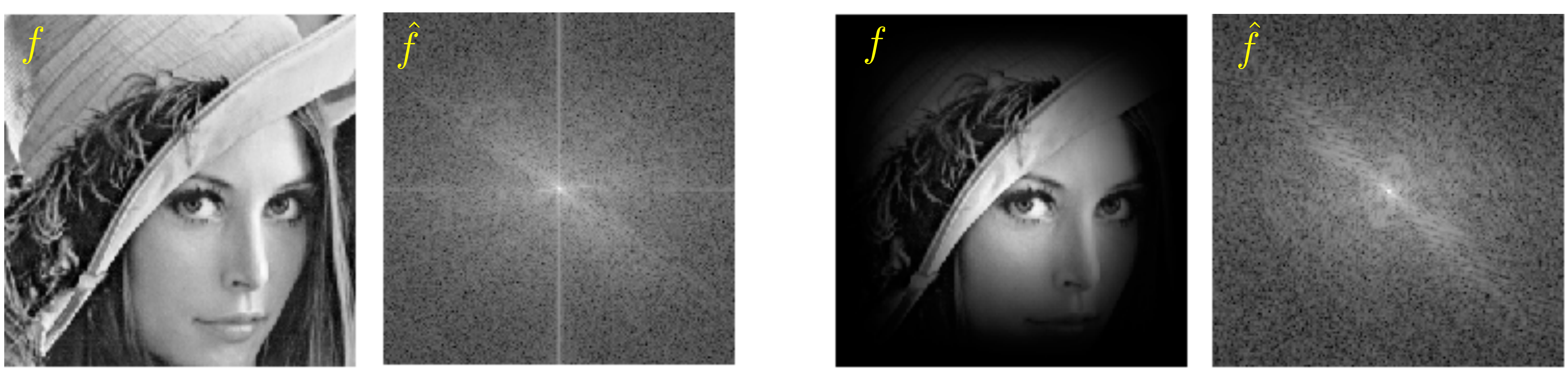
Frequency $(k_1, k_2) \in \{0, \ldots, N-1\}^2$

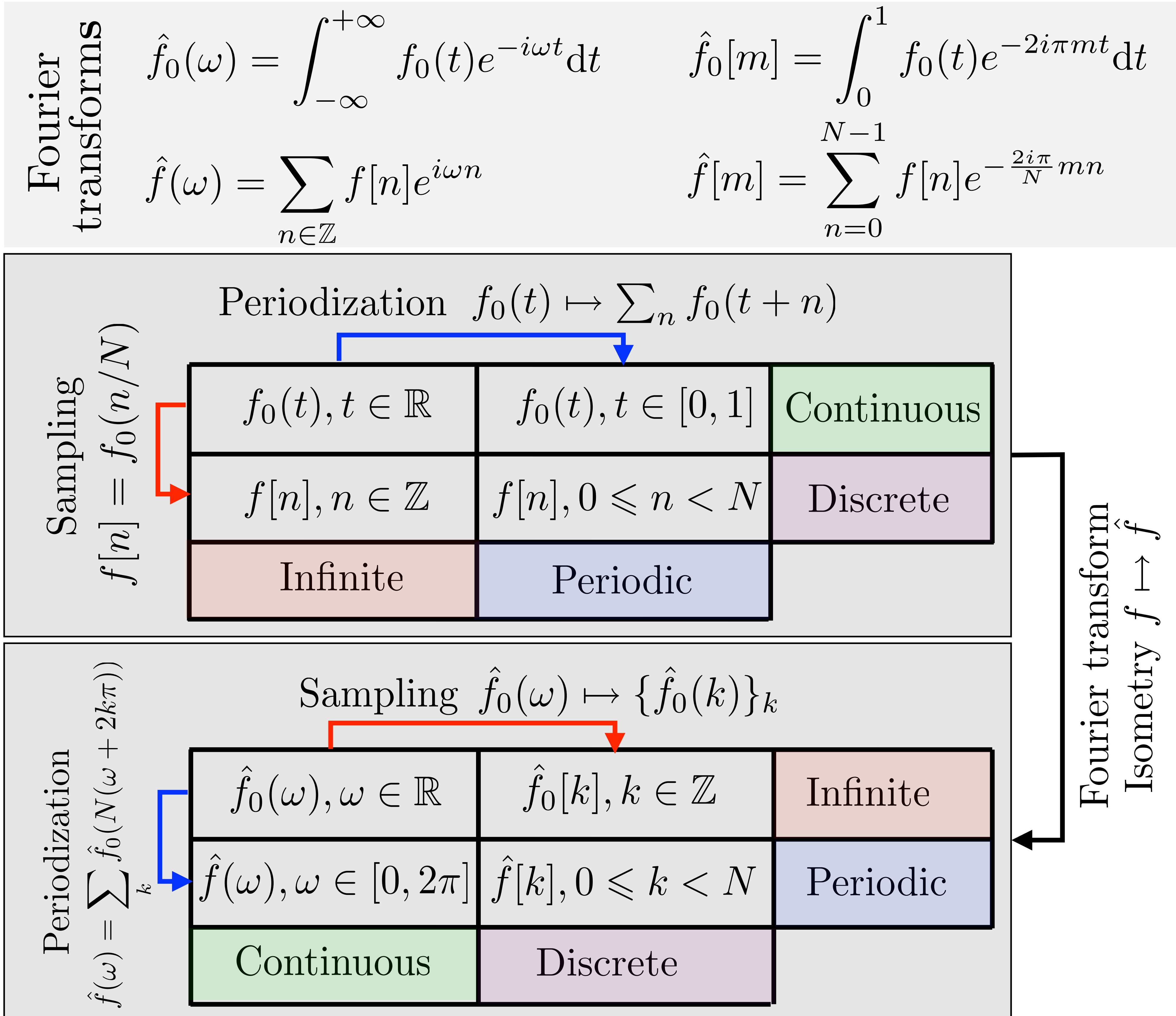


Fourier transform: $f \to \hat{f}$

$$\hat{f}[k_1,k_2] \overset{\text{def.}}{=} \langle f,\, e_{k_1,k_2} \rangle = \sum_{n_1,n_2} f[n_1,n_2] \bar{e}_{k_1,k_2}[n_1,n_2]$$

Fast Fourier Transform: $O(N^2 \log(N))$ operations

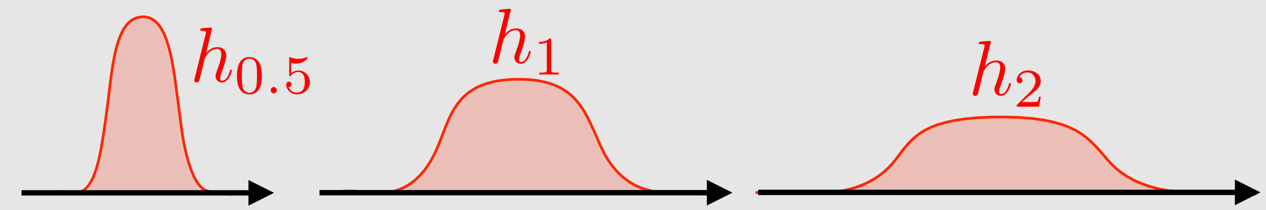




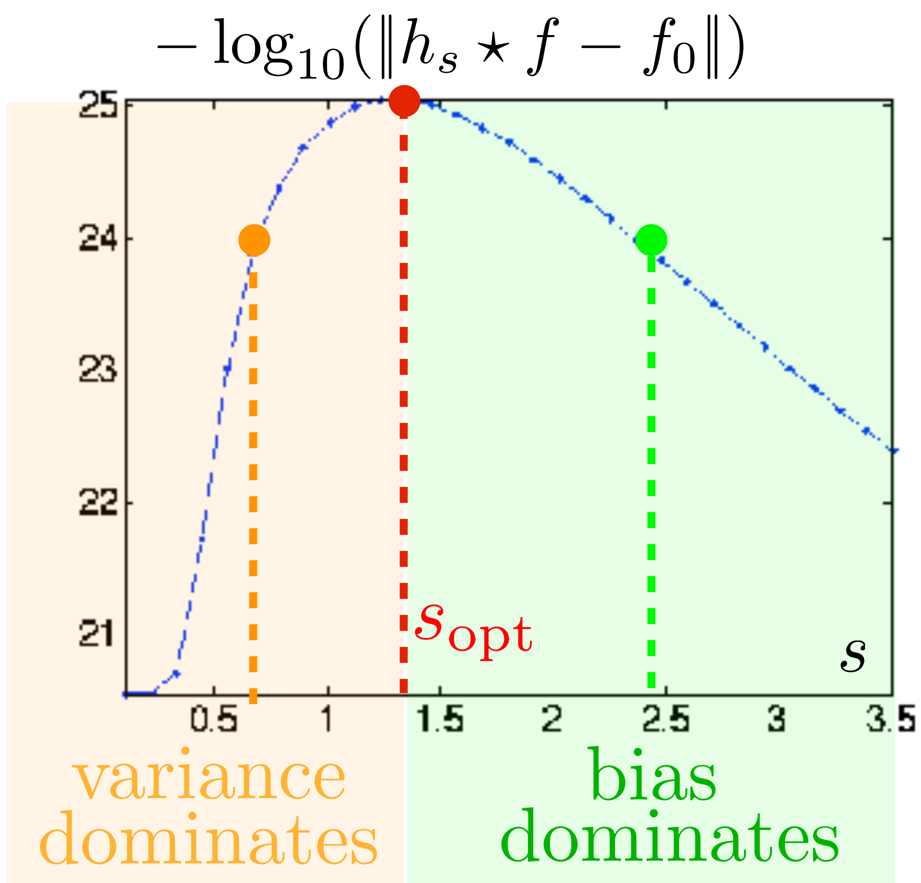
Fourier transforms
$\hat{f}_0(\omega) = \int_{-\infty}^{+\infty} f_0(t) e^{-i\omega t} \mathrm{d}t$
$\hat{f}_0[m] = \int_0^1 f_0(t) e^{-2i\pi m t} \mathrm{d}t$
$\hat{f}(\omega) = \sum_{n \in \mathbb{Z}} f[n] e^{i\omega n}$
$\hat{f}[m] = \sum_{n=0}^{N-1} f[n] e^{-\frac{2i\pi}{N} mn}$
Periodization $f_0(t) \mapsto \sum_n f_0(t+n)$
Sampling $f[n] = f_0(n/N)$
$f_0(t), t \in \mathbb{R}$
$f_0(t), t \in [0,1]$
Continuous
$f[n], n \in \mathbb{Z}$
$f[n], 0 \leqslant n < N$
Discrete
Infinite
Periodic
Fourier transform
Isometry $f \mapsto \hat{f}$
Sampling $\hat{f}_0(\omega) \mapsto \{\hat{f}_0(k)\}_k$
Periodization $\hat{f}(\omega) = \sum_k \hat{f}_0(N(\omega + 2k\pi))$
$\hat{f}_0(\omega), \omega \in \mathbb{R}$
$\hat{f}_0[k], k \in \mathbb{Z}$
Infinite
$\hat{f}(\omega), \omega \in [0, 2\pi]$
$\hat{f}[k], 0 \leqslant k < N$
Periodic
Continuous
Discrete

Data generation model: f_0 (unknown) $\xrightarrow{\text{noise}}$ f (noisy)

Dilated filter: $h_s = \frac{1}{s^{d/2}} h(\cdot/s)$



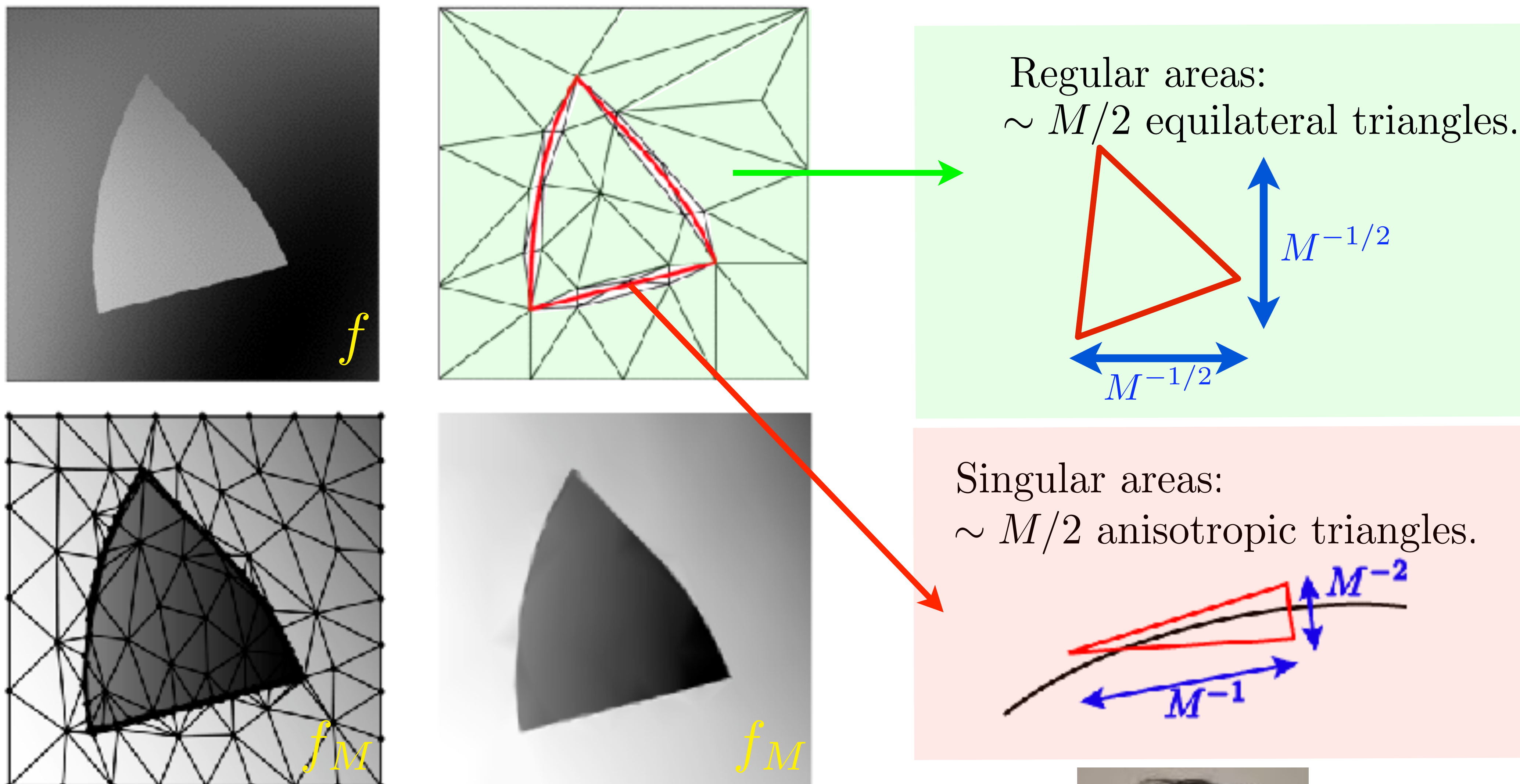
Linear invariant denoiser: $f \xrightarrow{\text{convolution}} f \star h_s \overset{\text{def.}}{=} \sum_k f[k] h_s[\cdot - k]$



$$\underbrace{\|h_s \star f - f_0\|}_{\text{error}} \leqslant \underbrace{\|h_s \star (f - f_0)\|}_{\text{variance}} + \underbrace{\|h_s \star f_0 - f_0\|}_{\text{bias}}$$

$s < s_{\text{opt}}$ $\quad$ $s = s_{\text{opt}}$ $\quad$ $s > s_{\text{opt}}$

Piecewise linear approximation using M triangles:

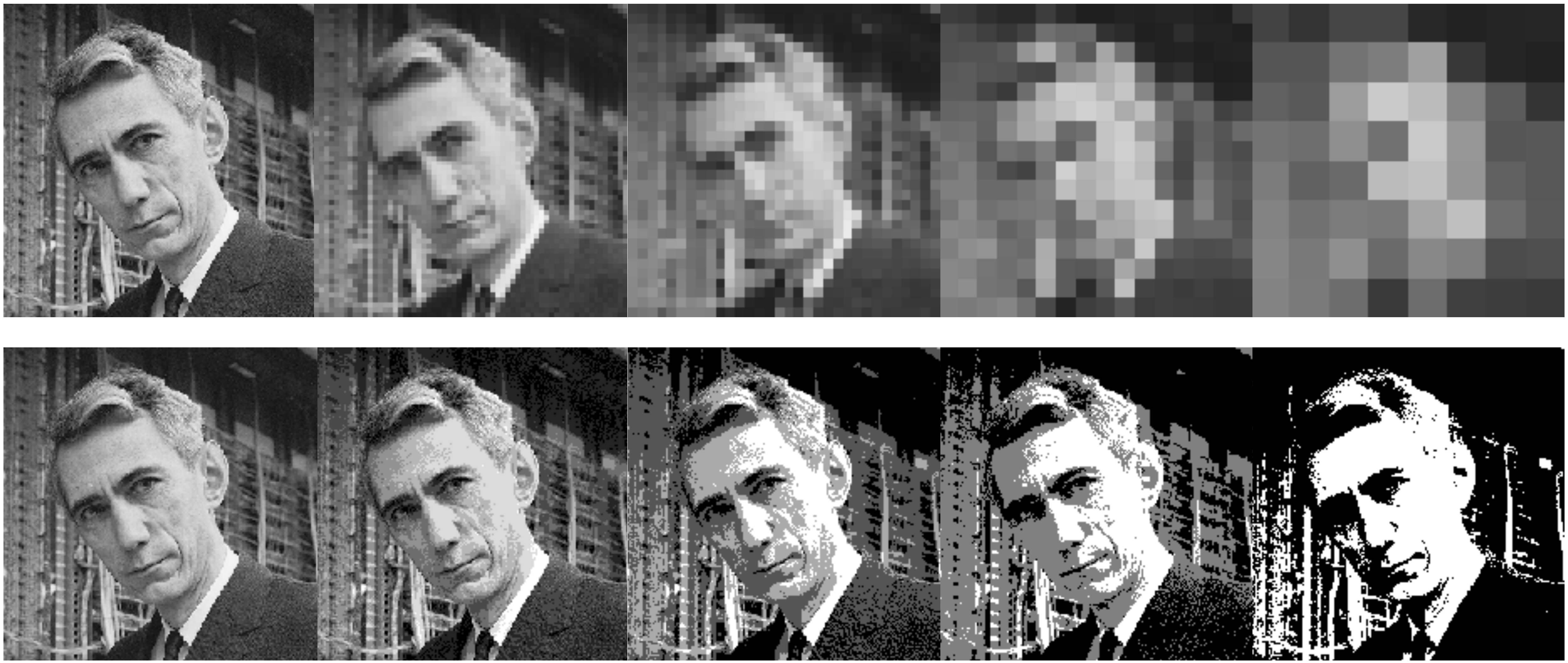


Theorem: If f is C^2 outside C^2 curves,

$$\|f - f_M\|^2 = O(M^{-2})$$

$\rightarrow$ to be compared with wavelets:

$$\|f - f_M\|^2 = O(M^{-1})$$



Thin plate spline: $\min_{f} \sum_{k=1}^{K} \|y_k - f(x_k)\|^2 + \lambda \|\Delta f\|^2$

Theorem: $\exists A \in R^{2\times 2}, b \in \mathbb{R}^2, (w_k \in \mathbb{R}^2)_k$

$$f(x) = Ax + b + \sum_{k=1} w_k \varphi(\|x - x_k\|) \qquad \varphi(r) = r^2 \log(r)$$

solve a linear system