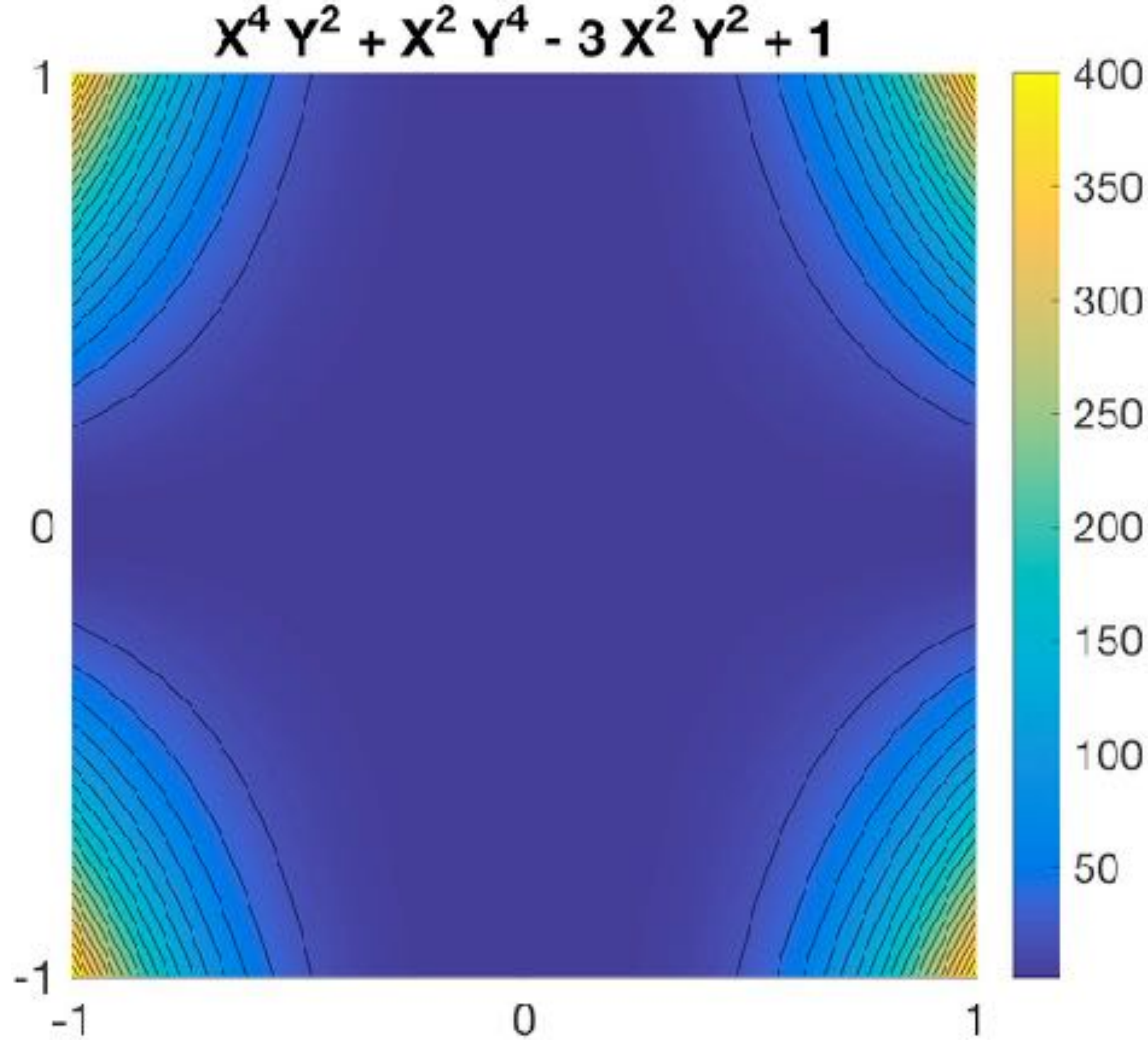
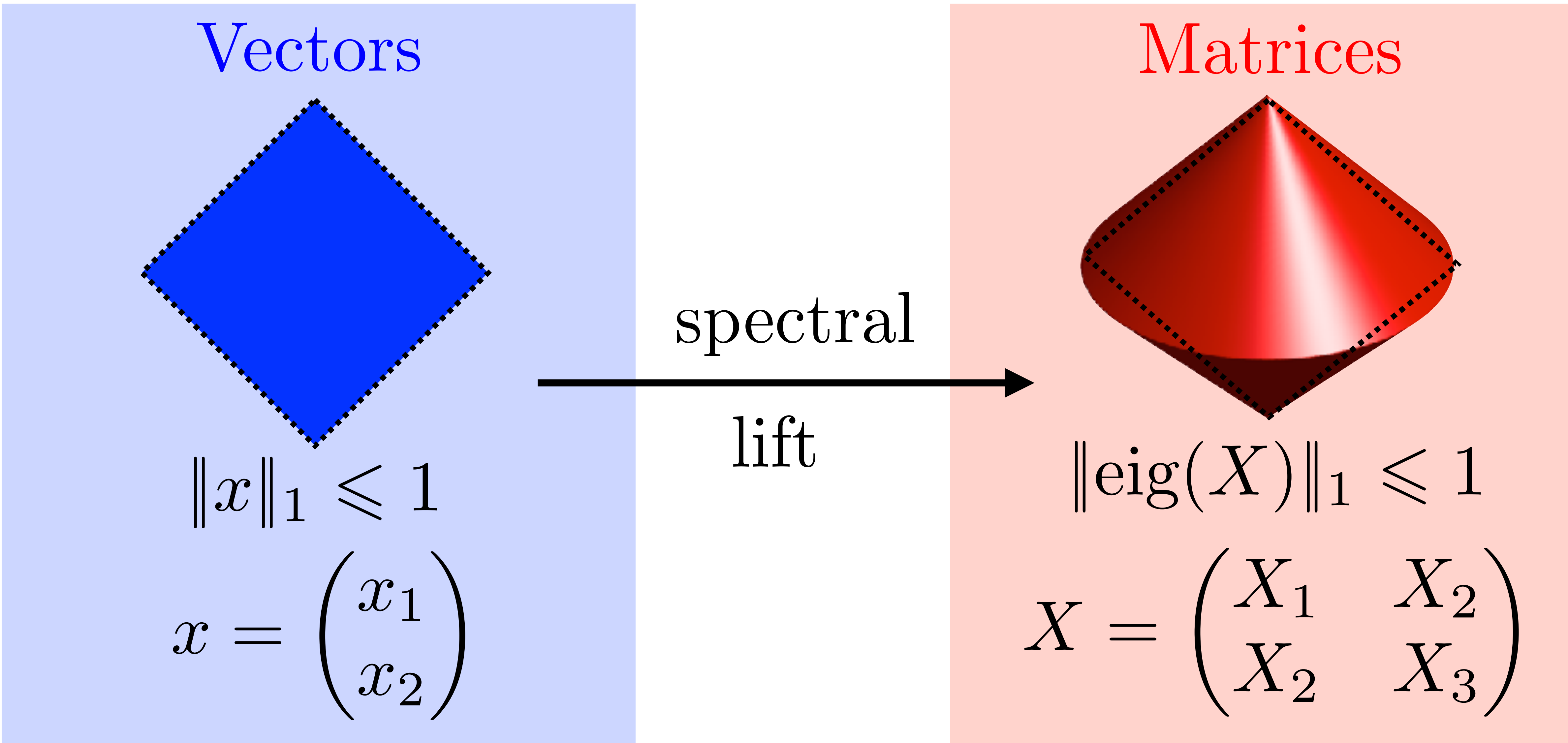
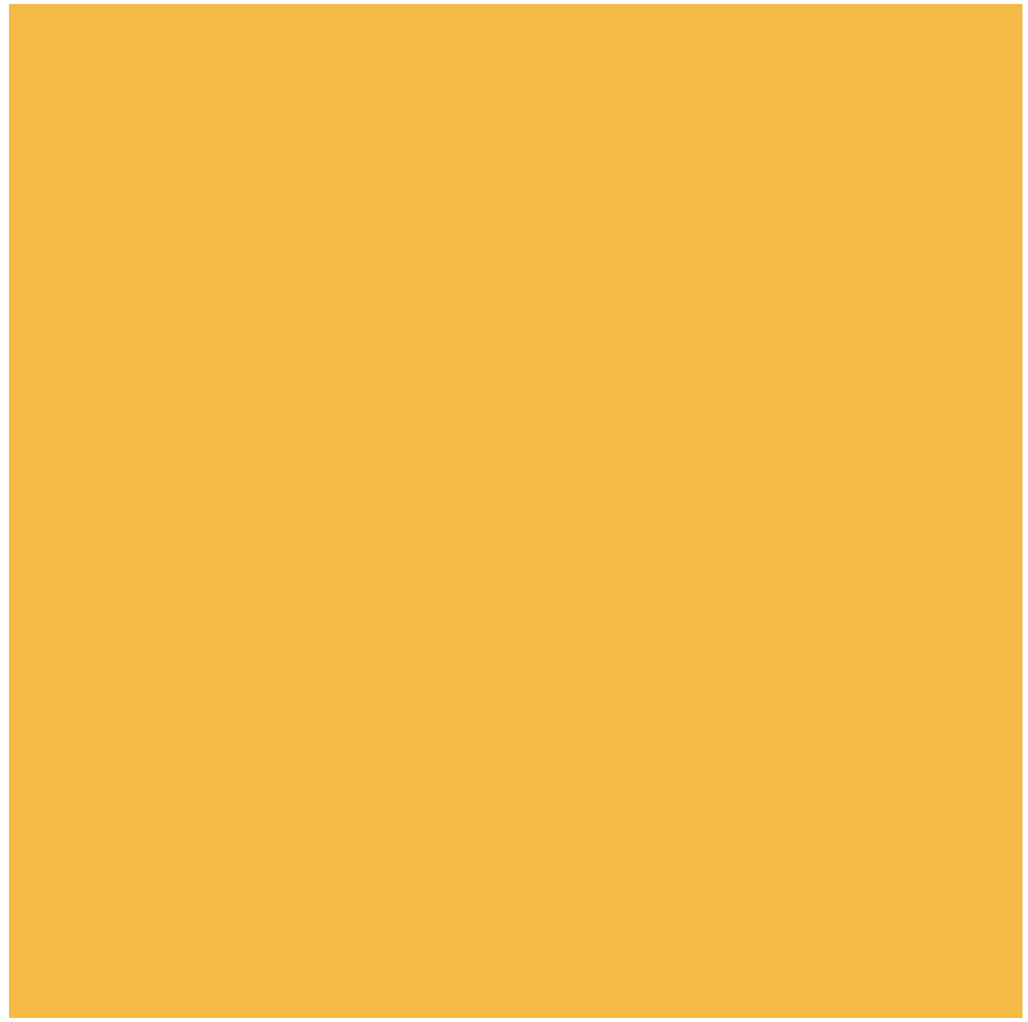
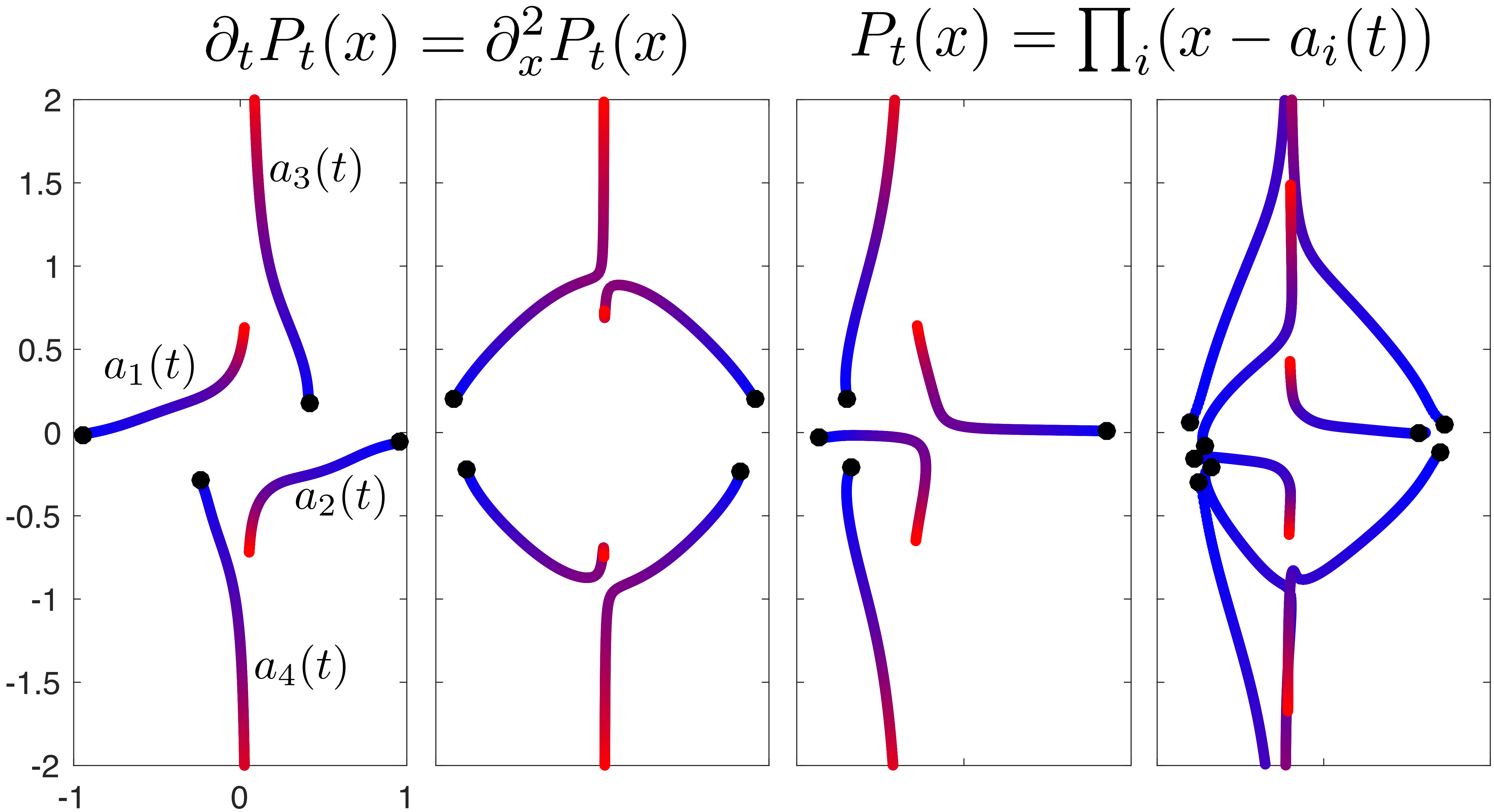
Matrix and Polynomials



$X^4 Y^2 + X^2 Y^4 - 3 X^2 Y^2 + 1$
1
0
-1
-1
0
1
400
350
300
250
200
150
100
50

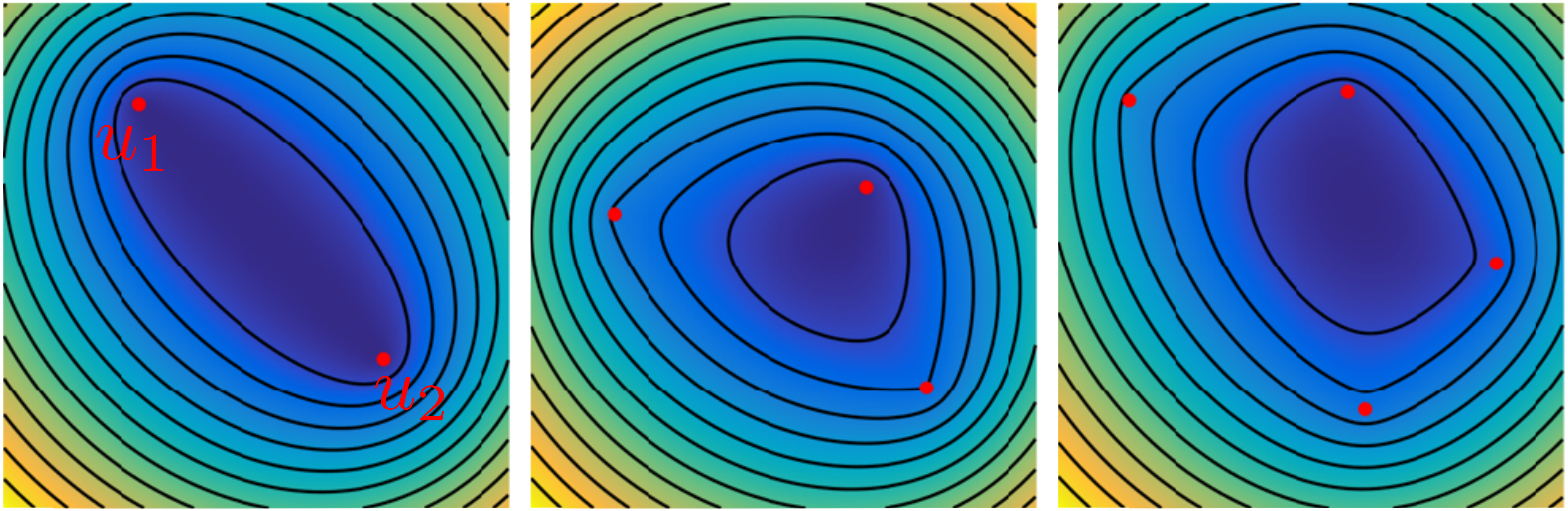

Vectors
Matrices
spectral
lift
$\|x\|_1 \leqslant 1$
$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$
$\|\mathrm{eig}(X)\|_1 \leqslant 1$
$X = \begin{pmatrix} X_1 & X_2 \\ X_2 & X_3 \end{pmatrix}$





$\partial_t P_t(x) = \partial_x^2 P_t(x)$
$P_t(x) = \prod_i (x - a_i(t))$
$a_1(t)$
$a_2(t)$
$a_3(t)$
$a_4(t)$
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1
0
1

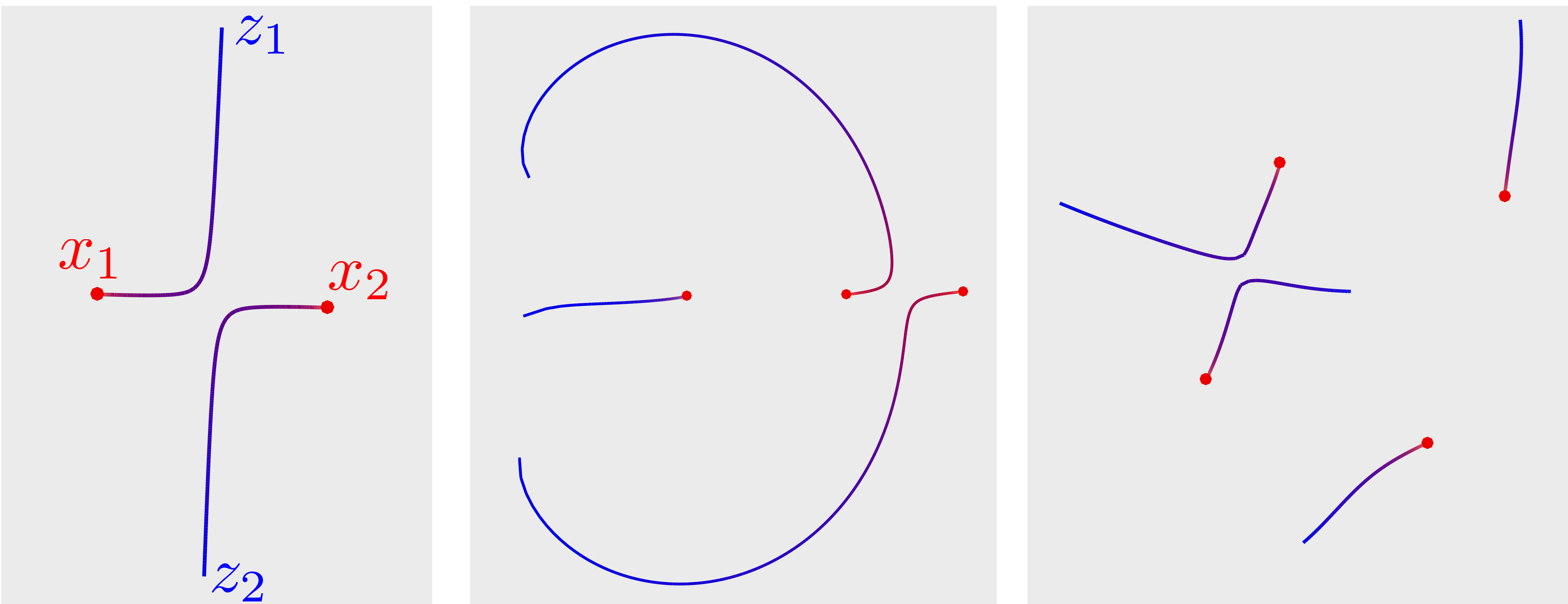
$$\left\{(x,y)\in\mathbb{R}^2\ ;\ \sum_{i=1}^{n}\sqrt{(x-u_k)^2+(y-v_k)^2}\leqslant t\right\}$$



$$z_i \longleftarrow z_i - \frac{P(z_i)}{\prod_{j\neq i}(z_i - z_j)}$$

Theorem:

$$(z_i)_i \longrightarrow \mathrm{Roots}(P)$$

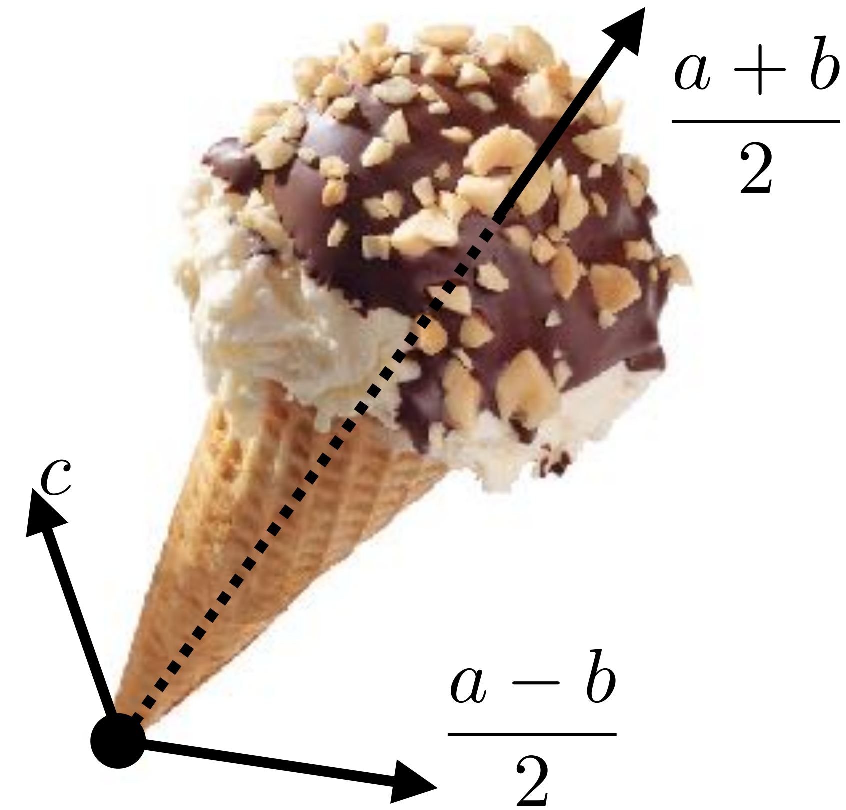


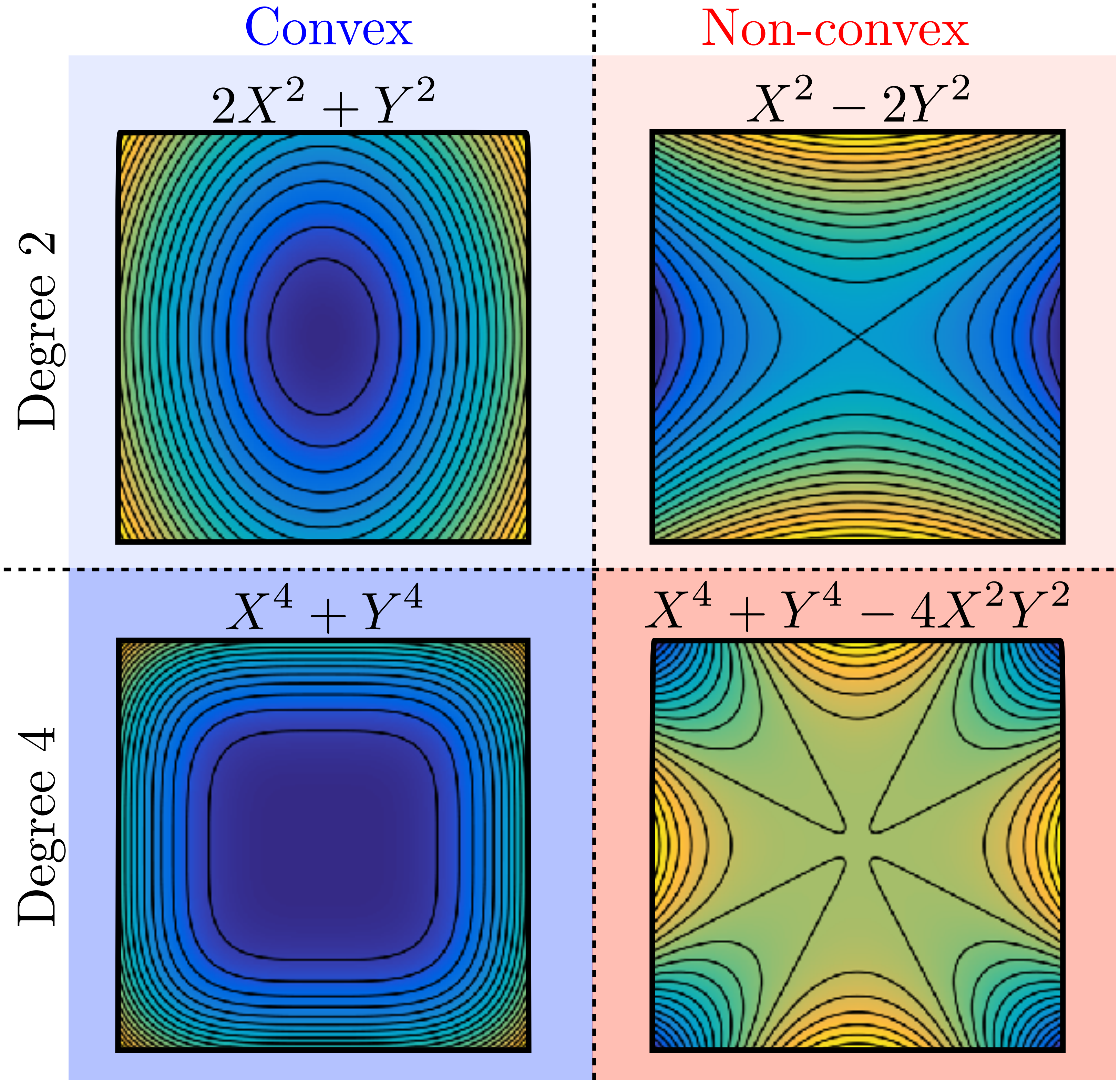
$$P(x) = \prod_i (x - x_i)$$

$$\left\{(a, b, c)\ ;\ \forall x, a + 2cx + bx^2 \geqslant 0\right\}$$

$$= \left\{(a, b, c)\ ;\ \begin{pmatrix} a & c \\ c & b \end{pmatrix} \succeq 0\right\}$$

$$= \left\{(a, b, c)\ ;\ \left(\frac{a-b}{2}\right)^2 + c^2 \leqslant \left(\frac{a+b}{2}\right)^2\right\}$$

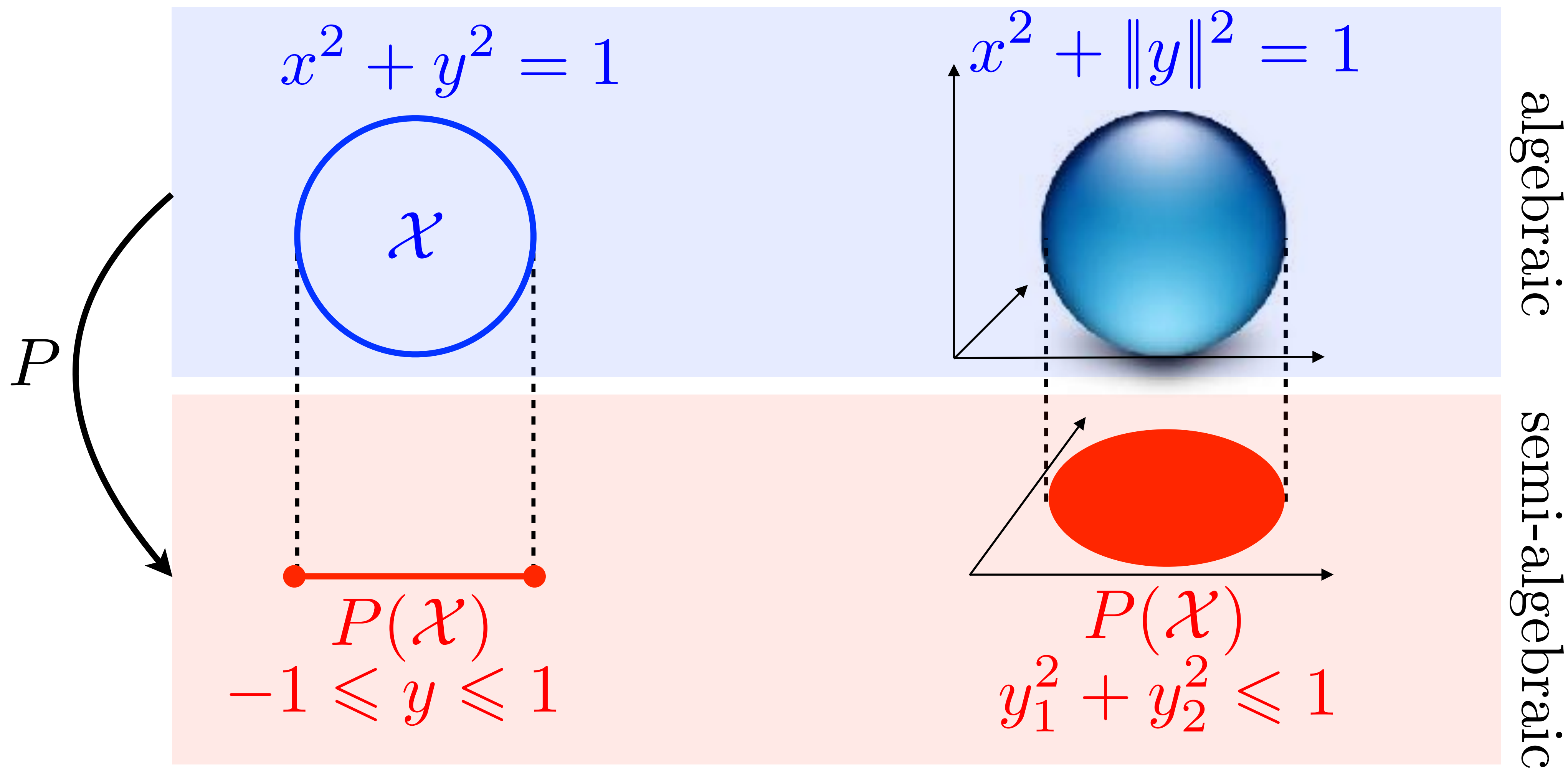




Convex
Non-convex
$2X^2 + Y^2$
$X^2 - 2Y^2$
Degree 2
$X^4 + Y^4$
$X^4 + Y^4 - 4X^2Y^2$
Degree 4

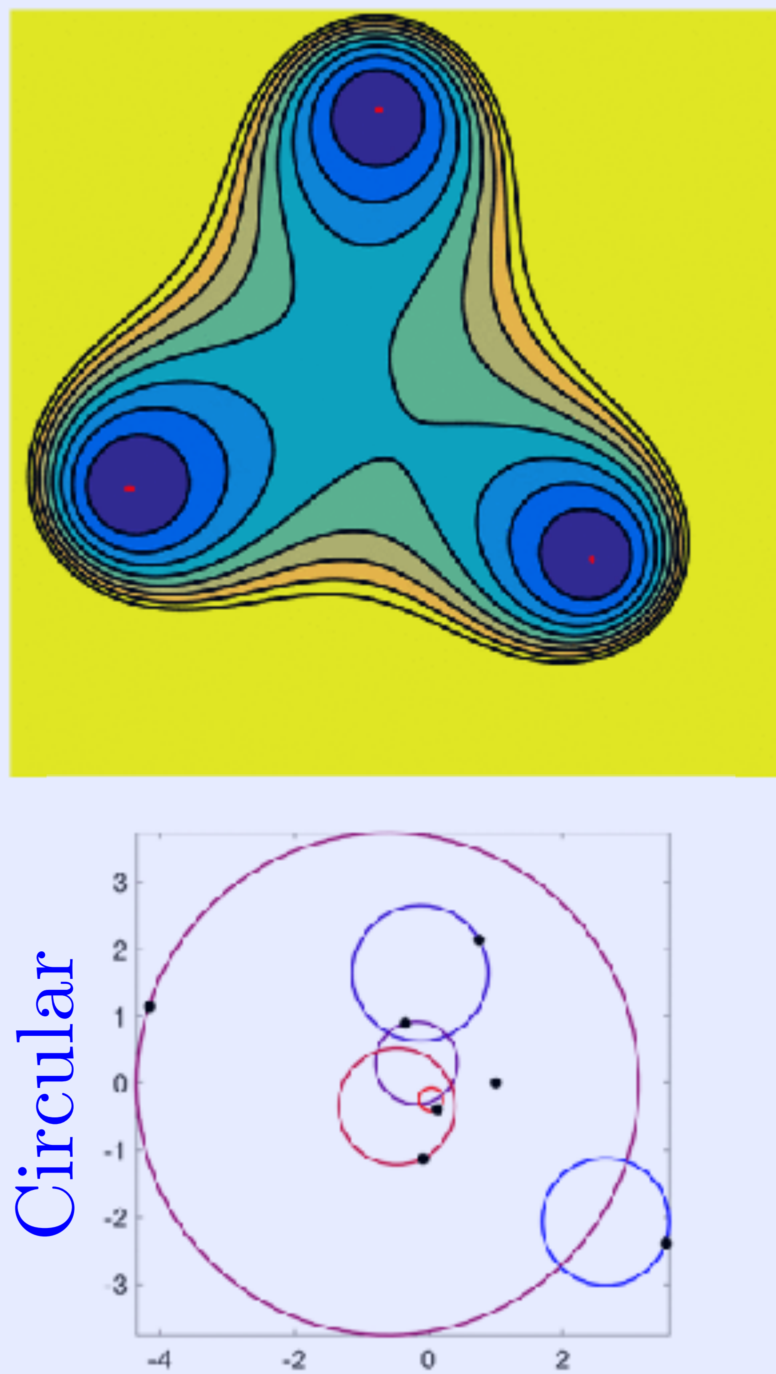
$\mathcal{X} \subset \mathbb{R}^n$ semi-algebraic $\Longrightarrow$ $P(\mathcal{X}) \in \mathbb{R}^{n-1}$ semi-algebraic.

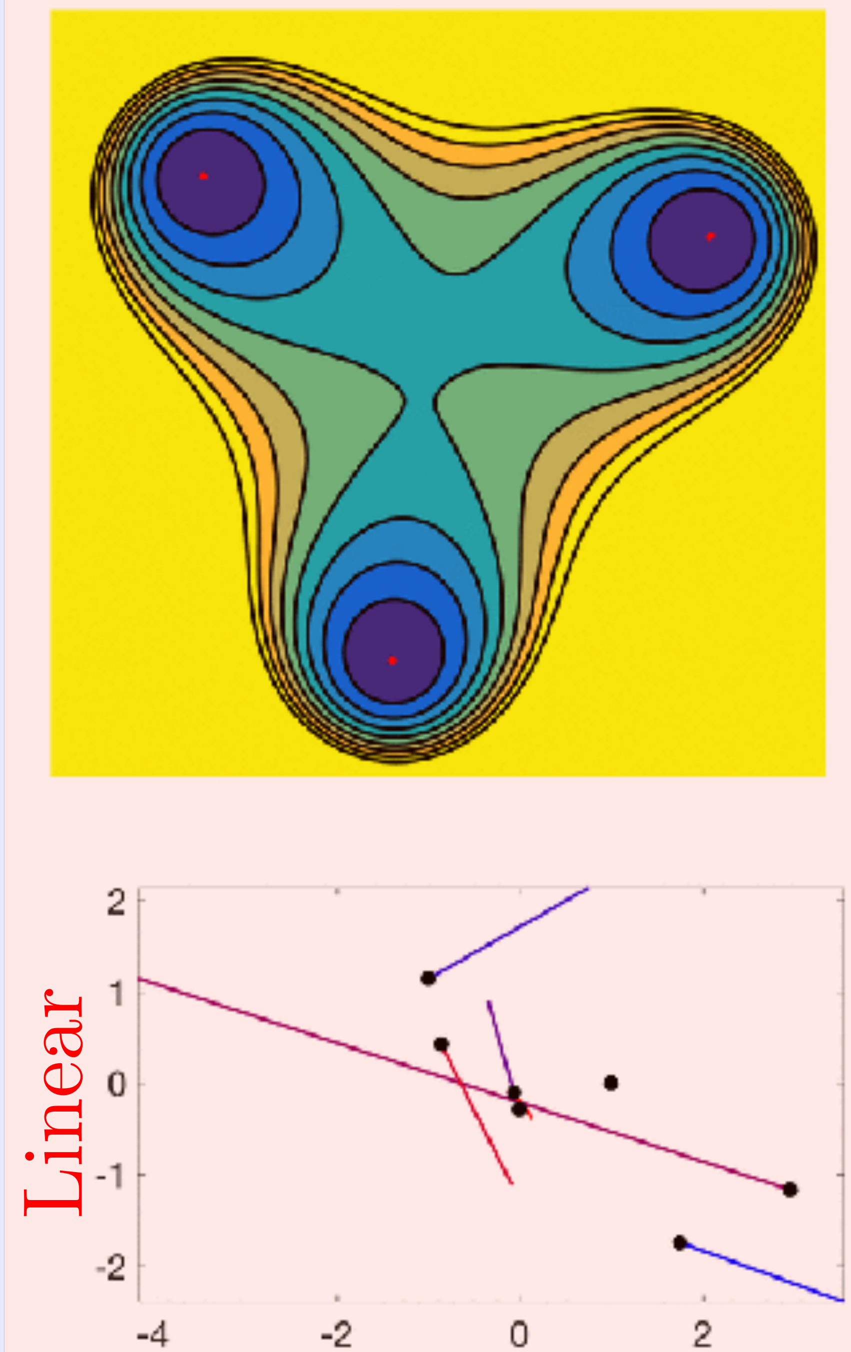
$$P : (x, y) \in \mathbb{R} \times \mathbb{R}^{n-1} \longmapsto y \in \mathbb{R}^{n-1}$$



$$(x_{i,t})_i = \text{roots}\left((1-t)\prod_i (x - x_{i,0}) + t \prod_i (x - x_{i,1})\right)$$

$$(x_{i,t})_i = \text{roots}\left(\frac{1+e^{2i\pi t}}{2}\prod_i (x - x_{i,0}) + \frac{1-e^{2i\pi t}}{2} \prod_i (x - x_{i,1})\right)$$





Simplex: $\Sigma_k = \{p \in \mathbb{R}_+^k \; ; \; \sum_i p_i = 1\}$

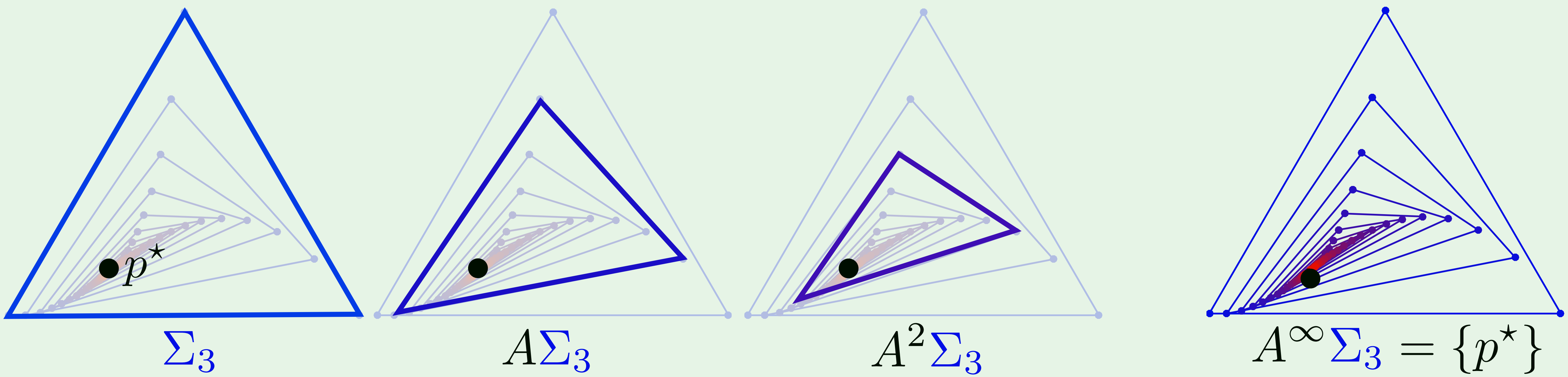
Stochastic matrix: $A \in \mathbb{R}_+^n$, $A^\top \mathbb{1}_k = \mathbb{1}_k$

$A : \Sigma_k \to \Sigma_k$

Theorem: [Perron-Frobenius]

If $A > 0$, $\exists! p^\star$, $Ap^\star = p^\star$.

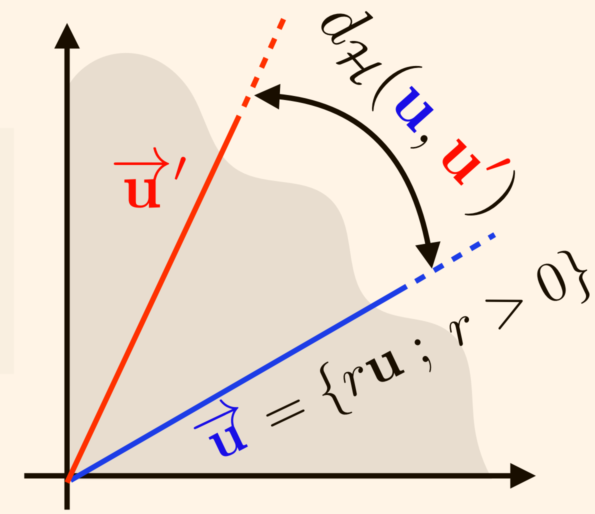
$\exists \rho \in [0, 1[$, $\|A^k p - p^\star\| \leqslant \rho^k$



Hilbert's projective metric:

$$\forall\,(\mathbf{u},\mathbf{u}') \in (\mathbb{R}^n_{+,*})^2, \quad d_{\mathcal{H}}(\mathbf{u},\mathbf{u}') \overset{\text{def.}}{=} \log \max_{i,i'} \frac{\mathbf{u}_i \mathbf{u}'_{i'}}{\mathbf{u}_{i'} \mathbf{u}'_i}.$$

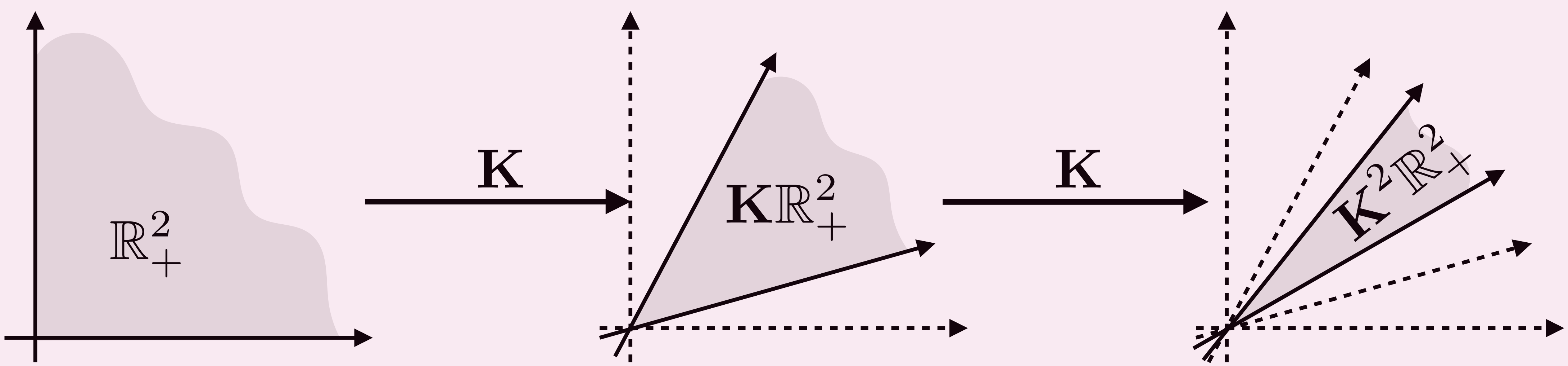
$d_{\mathcal{H}}$ is a distance on the set of rays $\overrightarrow{\mathbf{u}}$.

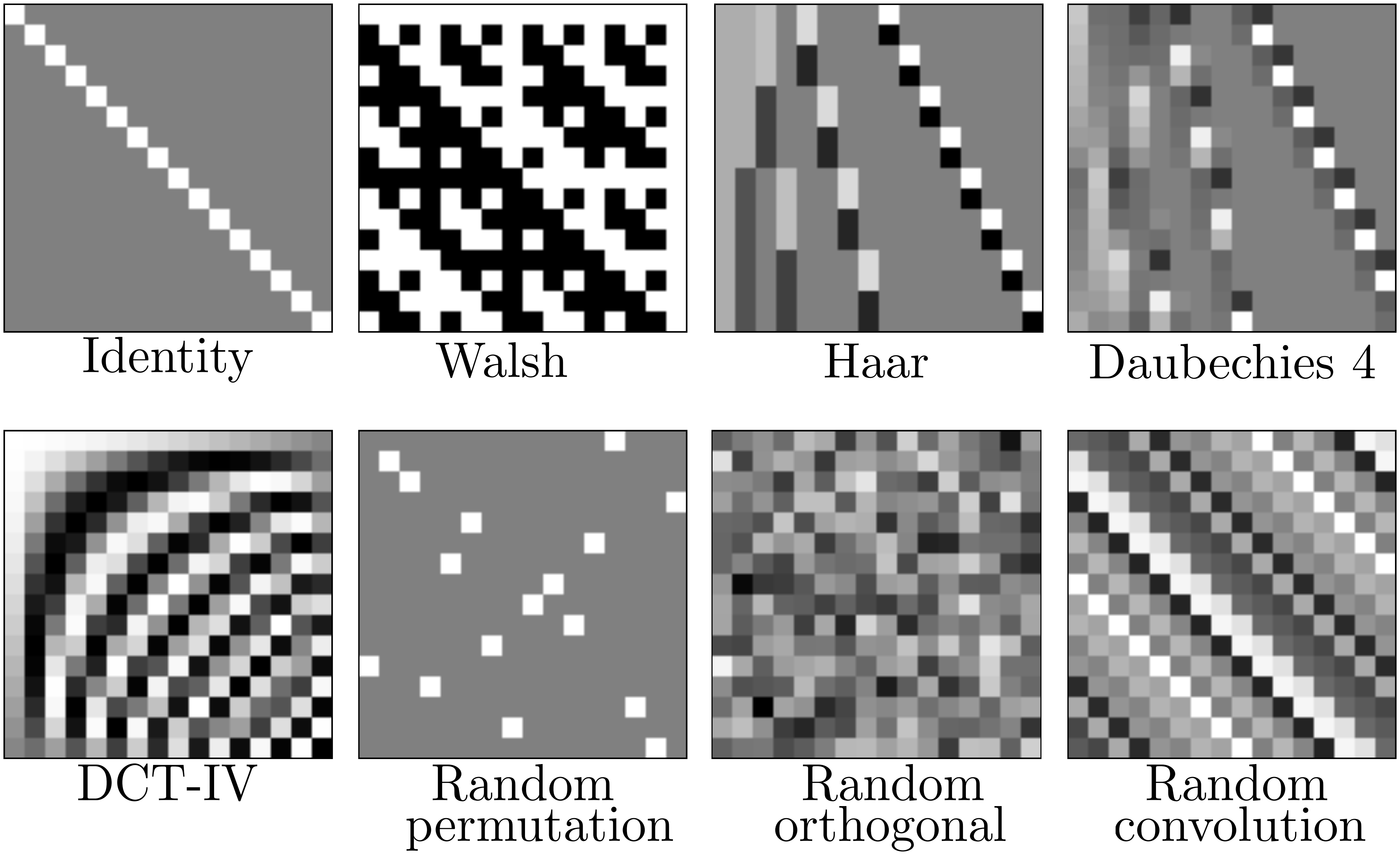


Birkhoff's contraction theorem:

Theorem 1.1. Let $\mathbf{K} \in \mathbb{R}^{n\times m}_{+,*}$, then for $(\mathbf{v},\mathbf{v}') \in (\mathbb{R}^m_{+,*})^2$

$$d_{\mathcal{H}}(\mathbf{K}\mathbf{v},\mathbf{K}\mathbf{v}') \leq \lambda(\mathbf{K}) d_{\mathcal{H}}(\mathbf{v},\mathbf{v}') \text{ where } \begin{cases} \lambda(\mathbf{K}) \overset{\text{def.}}{=} \frac{\sqrt{\eta(\mathbf{K})}-1}{\sqrt{\eta(\mathbf{K})}+1} < 1 \\ \eta(\mathbf{K}) \overset{\text{def.}}{=} \max_{i,j,k,\ell} \frac{\mathbf{K}_{i,k}\mathbf{K}_{j,\ell}}{\mathbf{K}_{j,k}\mathbf{K}_{i,\ell}}. \end{cases}$$




Identity
Walsh
Haar
Daubechies 4
DCT-IV
Random permutation
Random orthogonal
Random convolution

Cauchy-Binet formula:

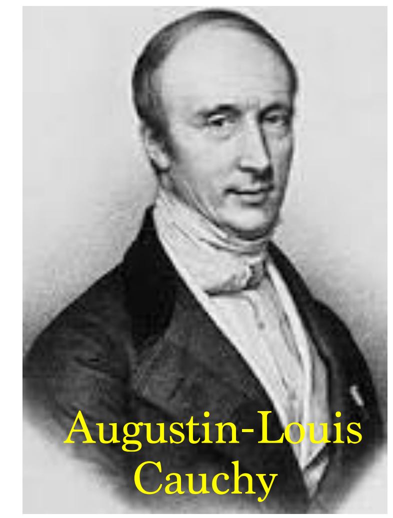
$$\det(AB) = \sum_{S \in \binom{[n]}{m}} \det(A_{[m],S}) \det(B_{S,[m]})$$

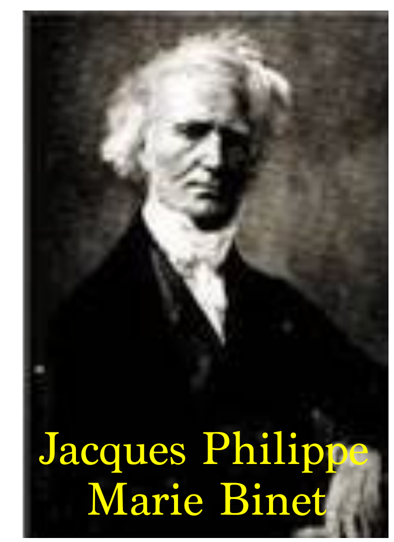
$$A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 0 & 2 \end{pmatrix}$$

$$\det(AB) = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} \cdot \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}$$

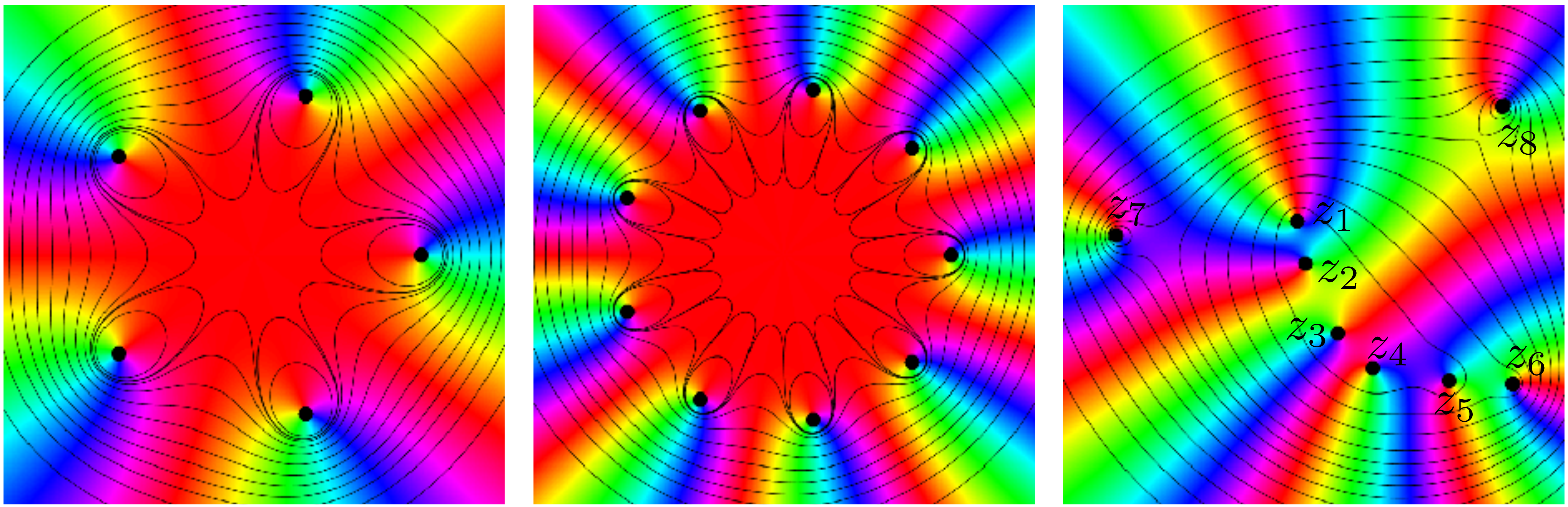
Binet–Cauchy identity: (case $m = 2$)

$$\left(\sum_{i=1}^{n} a_i c_i\right)\left(\sum_{j=1}^{n} b_j d_j\right) = \left(\sum_{i=1}^{n} a_i d_i\right)\left(\sum_{j=1}^{n} b_j c_j\right) + \sum_{1 \le i < j \le n} (a_i b_j - a_j b_i)(c_i d_j - c_j d_i)$$





color=Arg($P(z)$) iso-curves=$|(P(z))|$



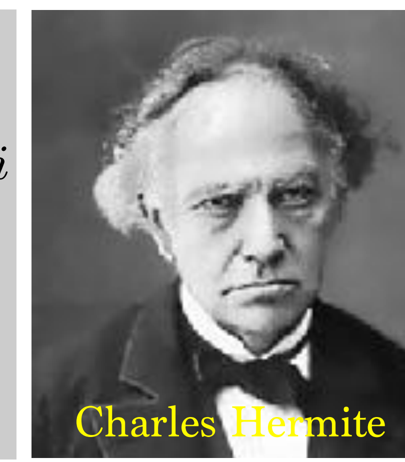
$P(z) = z^5 - 1$ $P(z) = z^{10} - 1$ $P(z) = \prod_i (z - z_i)$

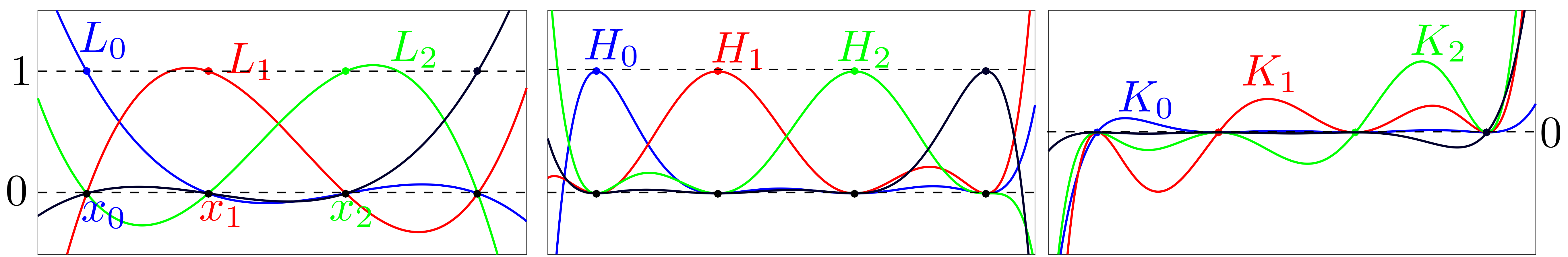
Theorem: $\forall$ distinct $(x_i)_{i=0}^n$, $\forall\,(a_i)_i$, $\exists! P \in \mathbb{R}_n[X]$, $\forall i, P(x_i) = a_i$.

$$P(x) = \sum_i a_i L_i(x) \qquad L_i(x) = \frac{\prod_{j\neq i}(x - x_j)}{\prod_{j\neq i}(x_i - x_j)}$$

Theorem: $\forall$ distinct $(x_i)_{i=0}^n$, $\forall\,(a_i, b_i)_i$, $\exists! P \in \mathbb{R}_{2n+1}[X]$, $\forall i$, $\begin{cases} P(x_i) = a_i \\ P'(x_i) = b_i \end{cases}$

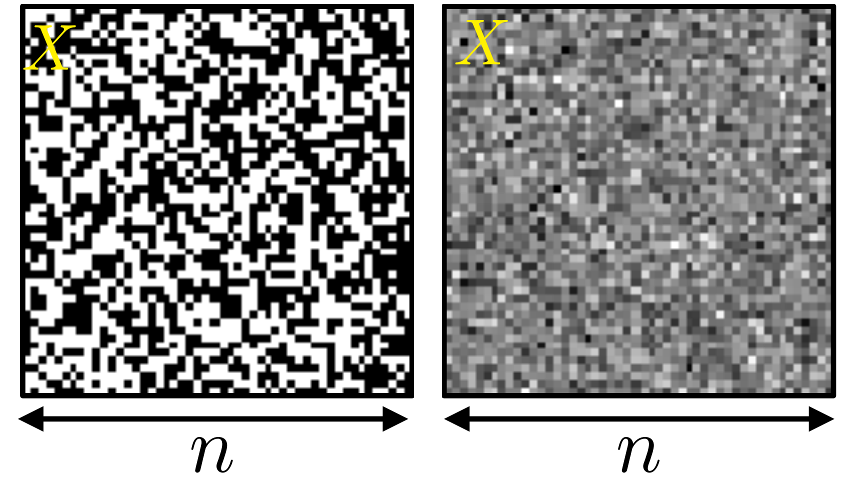
$$P(x) = \sum_i a_i H_i(x) + b_i K_i(x) \qquad \begin{aligned} K_i(x) &= L_i(x)^2 (x - x_i) \\ H_i(x) &= L_i(x)^2 (1 - 2L_i'(x_i)(x - x_i)) \end{aligned}$$





Random matrices:

$(X_{i,j})_{i,j=1}^{n}$ i.i.d. with $\mathbb{E}(X_{i,j}) = 0, \mathbb{E}(X_{i,j}^2) = \frac{1}{n}$.



Empirical eigenvalues distribution: $\mu_n = \sum_{k=1}^{n} \delta_{\lambda_k(X)}$

Theorem: [Tao, Vu, 2010] $\mathbb{P}(\mu_n \xrightarrow{n\to+\infty} \frac{1_D}{\pi}) = 1$